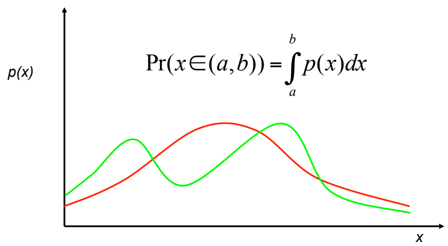
Uncertainty and Error Propagation



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Where We Are

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Last Time

- RANSAC
- Structure from Motion
- Sensor Fusion
- Sensor Error
  - Probability review
  - Measuring Error
  - Propagating error

This Time

- Statistics Review
- Error
- Error Propagation
- How a DC motor works
- Building Motors
- Choosing groups!

Uncertainty Representation

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- Sensing is always related to uncertainties.
  - How can uncertainty be represented or quantified?
  - How does it propagate – what's the uncertainty of a function of uncertain values?
  - How do uncertainties combine if different sensor reading are fused?
  - What is the merit of all this for robotics?

State Estimation

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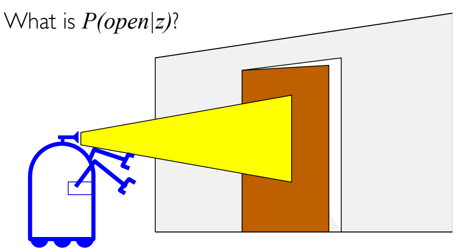
- Suppose a robot obtains measurement z
  - z = vision + edge detection
- What is $P(open|z)$?



Statistics Review

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- Expected value of a real-valued random variable X with density f(x):
  - E[X] = ∫x f(x)
- Expected value of a discrete-valued random variable X with distribution P(x):
  - E[X] = Σx P(x)
  - Suppose X corresponds to outcome of die roll
  - E[X] = 1 * 1/6 + 2 * 1/6 + 3 * 1/6 + 4 * 1/6 + 5 * 1/6 + 6 * 1/6
  - E[X] = 1/6 * (1 + 2 + 3 + 4 + 5 + 6) = 3.5
- If random variables X1 and X2 are independent, E[X1* X2] = E[X1] * E[X2]

Statistics Review

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- Variance: how far a set of numbers is spread out.
  - $E[(x - \mu)^2] = \int x^2 f(x) - \mu^2$
  - recall μ is the mean value
- If the variables are *correlated*, then we have *covariance*
- Covariance
  - Given two random variables, X1 and X2
  - $E[(X1 - \mu_{X1})(X2 - \mu_{X2})]$
  - What happens in the following case?
    - When X1 is above its mean, X2 tends to be below its mean
    - When X1 is above its mean, X2 tends to be way above its mean

Combining Evidence

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- Suppose our robot obtains another observation z_2.
- How can we integrate this new information?
- More generally, how can we estimate $P(x| z_1...z_n)$?

Recursive Bayesian Updating

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$$P(x \mid z_1,\dots,z_n) = \frac{P(z_n \mid x, z_1,\dots,z_{n-1})\, P(x \mid z_1,\dots,z_{n-1})}{P(z_n \mid z_1,\dots,z_{n-1})}$$

Markov assumption: z_n is independent of $z_1,...,z_{n-1}$ if we know x

$$P(x \mid z_1,\dots,z_n) = \frac{P(z_n \mid x)\, P(x \mid z_1,\dots,z_{n-1})}{P(z_n \mid z_1,\dots,z_{n-1})}$$

$$= \eta\, P(z_n \mid x)\, P(x \mid z_1,\dots,z_{n-1})$$

$$= \eta_{1\dots n} \prod_{i=1\dots n} P(z_i \mid x)\, P(x)$$

$P(B|A)$: probability of B given A

Second Measurement

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- $P(z_2|open) = 0.5$ $\quad P(z_2|\neg open) = 0.6$
- $P(open|z_1)=2/3$

$$P(open \mid z_2, z_1) = \frac{P(z_2 \mid open)\, P(open \mid z_1)}{P(z_2 \mid open)\, P(open \mid z_1) + P(z_2 \mid \neg open)\, P(\neg open \mid z_1)}$$

$$= \frac{\frac{1}{2}\cdot\frac{2}{3}}{\frac{1}{2}\cdot\frac{2}{3}+\frac{3}{5}\cdot\frac{1}{3}} = \frac{5}{8} = 0.625$$

z_2 gives higher probability that the door is open.

Error and Accuracy

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- Error: Difference between sensor output and true value

$$error = m\text{-}v \begin{cases} m = \text{measured value} \\ v = \text{true value} \end{cases}$$

- Accuracy: unitless measure

$$\left(accuracy = 1 - \frac{m - v}{v}\right)$$

error

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Precision (But Not Recall)

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- Precision: Reproducibility of sensor results
- A distribution of error can be characterized by:
  - Mean error: μ
  - Standard deviation: σ
  - How similar are two outputs from the same **test**?
    - Same sensor, same environment …

$$precision = \frac{range}{\sigma}$$

- Has other meanings in actuation *and* cognition

Statistical Representation of Error

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- Error: the difference between **measured** and **true** value
- How can we treat *sensing* as *estimation*?
- **X**: random variable representing actual value
  - E.g., "distance = 4 meters"
- **E[X]**: **estimate** of the true value
- Given **n** sensor readings $(\varrho_1, \varrho_2, \dots, \varrho_n)$
- $E[X] = g(\varrho_1, \varrho_2, \dots, \varrho_n)$

Representation of Uncertainty

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- Specific errors usually unknown, but…
- Errors exist on a spectrum:

  Deterministic ⟷ Non-deterministic (random)

- Some errors are **consistent** for some circumstances, and can be **characterized**. These are more deterministic.
- A probability density function gives a probability density f(x) for any x in X.

Representing Uncertainty

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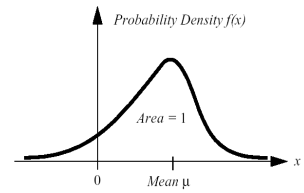
- Sensing as estimation problem:

  true (unknown) value = X
  estimate of value = $E[X]$

- Given n measurements with values : $\sigma_{[1\text{-}n]}$



Uncertainty Representation (2)

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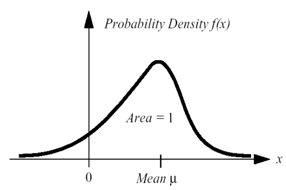
Area under curve = 1:

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

sum of all possible probability values.



Mean:

$$\mu = E[X] = \int_{-\infty}^{\infty} xf(x)dx$$

…if we measure X infinite times and average the values we see.

Variance:

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx$$

The "width" of possible values X might take.

Gaussian Distribution

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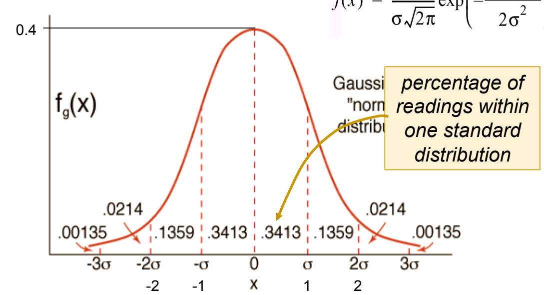
$\mu = 0$ and $\sigma = 1$

formula for Gaussian

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$



Error Distributions

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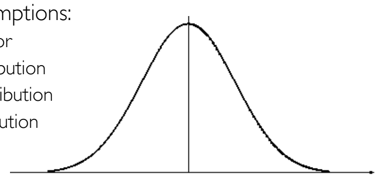
- Random errors: behavior of sensors modeled by some probability distribution
- Causes and behavior of error usually unknown
  - So what do we do?
- Simplifying assumptions:
  - Zero-mean error
  - Unimodal distribution
  - Symmetric distribution
  - Gaussian distribution



Simplifying Assumptions

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- Important to remember assumptions are wrong!

  Examples

- Sonar (ultrasonic) sensor more likely to **over**estimate distance in real environment
- Is therefore not symmetric
  - Might be better modeled by two modes:
    - Mode for the case that the signal returns directly
    - Mode for the case that the signals returns after reflections
- Stereo vision system might not correlate images
  - Results that make no sense at all

Error Propagation

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- How do we combine a series of uncertain measurements?
  - (Basically the usual case for sensing)
- Propagation of uncertainty (or propagation of error)
- **Fuse** a sequence of readings into a single value

Error Propagation Law

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- The effect of *variables' uncertainty* on the *uncertainty of a function that depends on them.*

  Absolute error Δx

- Error on some quantity, Δx, is given as

  Standard deviation: the positive square root of variance, σ^2

- With a probability distribution, can find confidence limits
  - How sure are we of our estimate?

The Error Propagation Law

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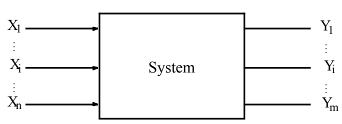

- Error propagation in a multiple-input multi-output system with *n* inputs and *m* outputs.

$$Y_j = f_j(X_1 \ldots X_n)$$

Error Propagation Law

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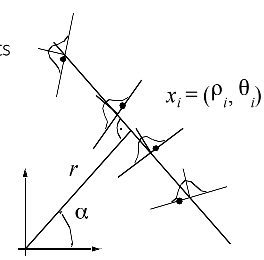
- Imagine extracting a line based on point measurements with uncertainties.
- The model parameters ρ_i (length of the perpendicular) and θ_i (its angle to the abscissa) describe a line uniquely.



- The question:
  - What is the uncertainty of the extracted line knowing the uncertainties of the measurement points that contribute to it?

The Error Propagation Law

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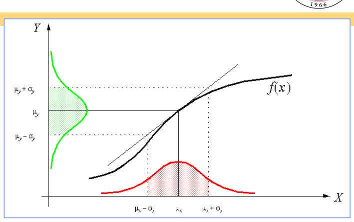
- One-dimensional case of a nonlinear error propagation problem
- It can be shown, that the output covariance matrix C_Y is given by the error propagation law:

$$C_Y = F_X C_X F_X^T$$



- where
  - C_X: covariance matrix representing the input uncertainties
  - C_Y: covariance matrix representing the propagated uncertainties for the outputs.
  - F_X: is the **Jacobian** matrix defined as:

$$F_X = \nabla f = \left[\nabla_X \cdot f(X)^T\right]^T = \begin{bmatrix} f_1 \\ \vdots \\ f_m \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial X_1} & \cdots & \frac{\partial}{\partial X_n} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial X_1} & \cdots & \frac{\partial f_1}{\partial X_n} \\ \vdots & \cdots & \vdots \\ \frac{\partial f_m}{\partial X_1} & \cdots & \frac{\partial f_m}{\partial X_n} \end{bmatrix}$$

  - which is the transposed of the gradient of f(X).