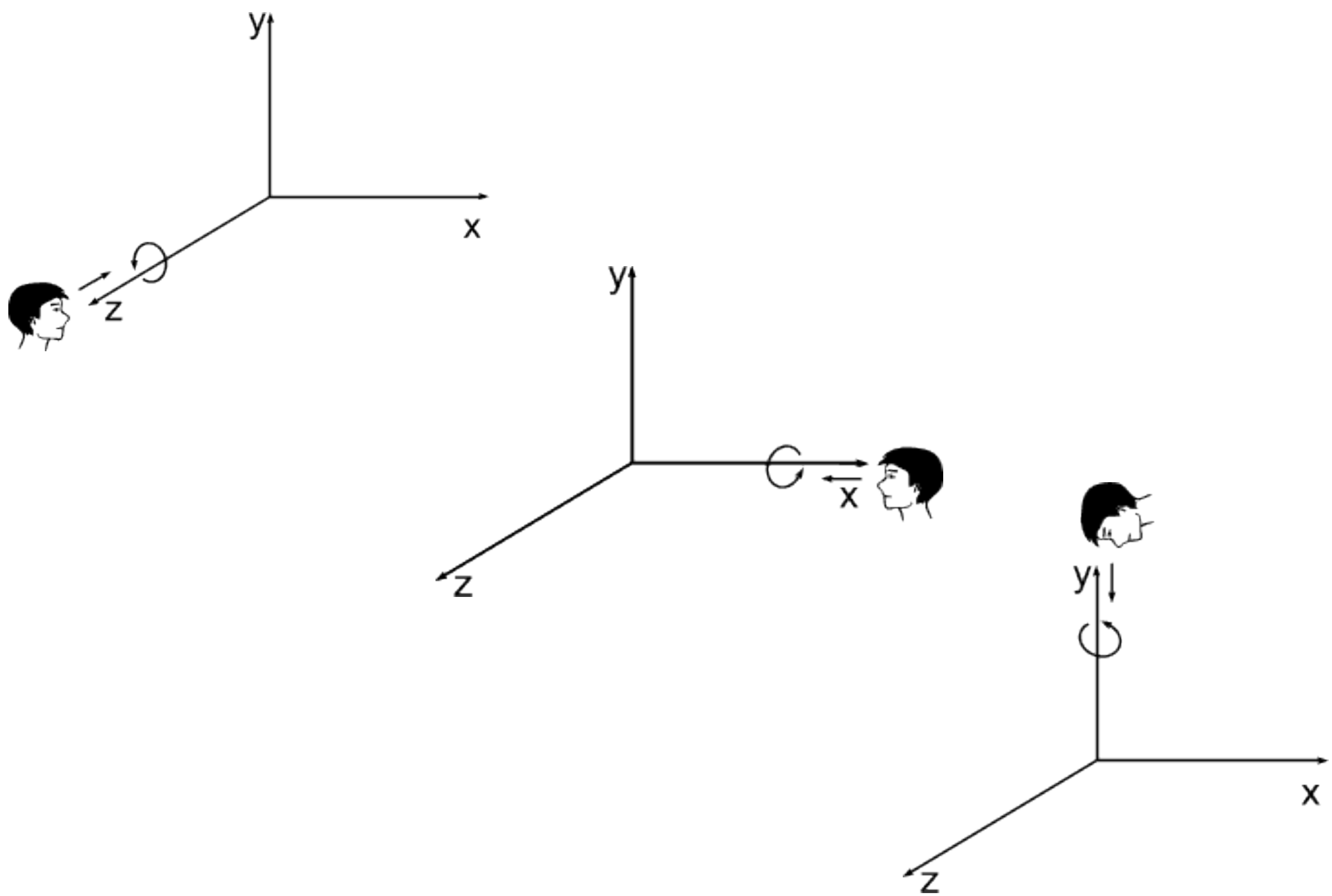
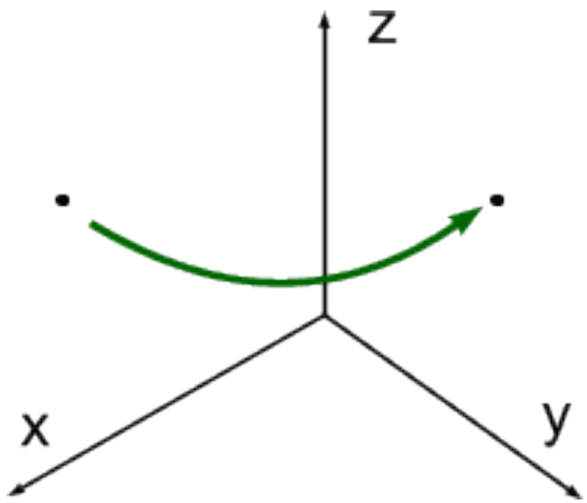
How 3D rotation works

Some slides courtesy of Dr. Penny Rheingans

Positive 3D Rotation



Rotation about Z axis



- Like 2D

  x' = x cosθ - y sinθ

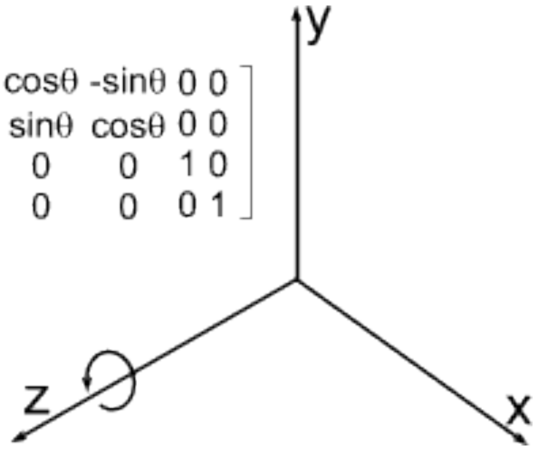
  y' = x sinθ + y cosθ

  z' = z

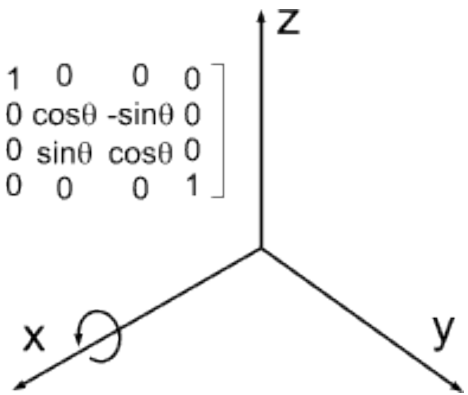
$$T = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation about Coordinate Axes

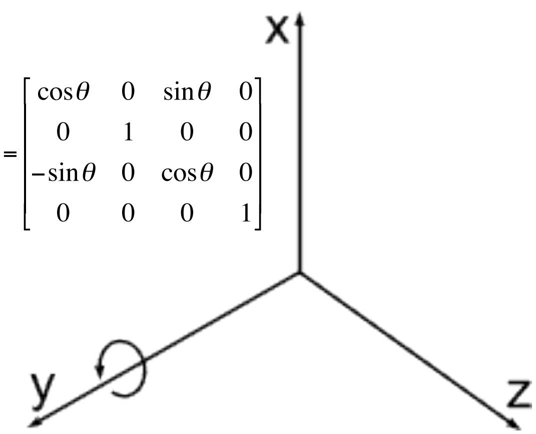
$$T=\begin{bmatrix}\cos\theta & -\sin\theta & 0 & 0\\ \sin\theta & \cos\theta & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$



$$T=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & \cos\theta & -\sin\theta & 0\\ 0 & \sin\theta & \cos\theta & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

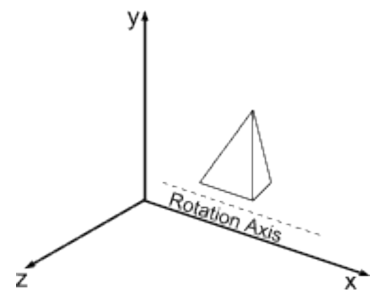


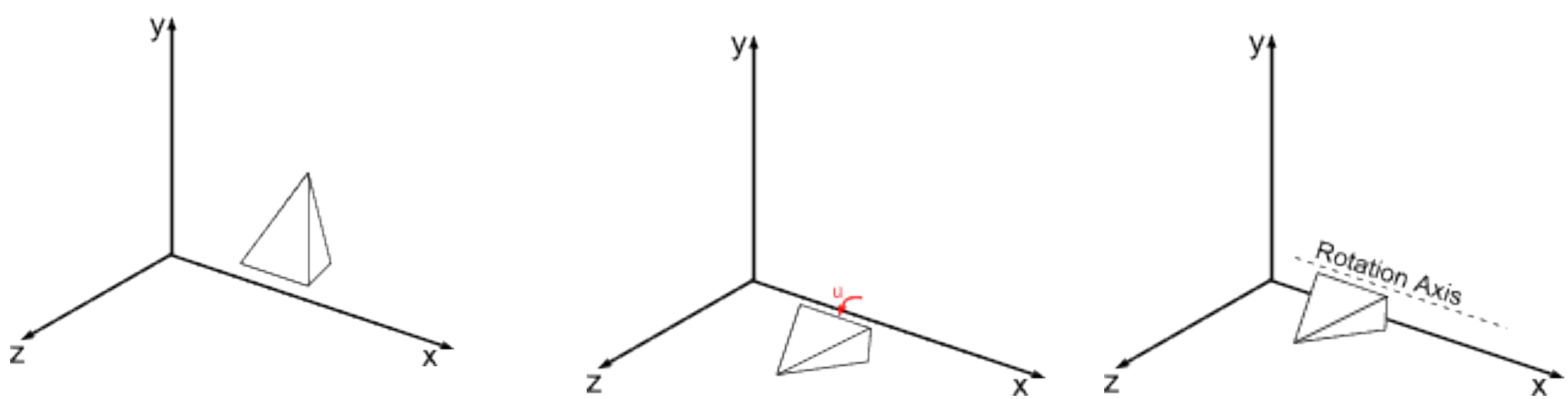
$$T=\begin{bmatrix}\cos\theta & 0 & \sin\theta & 0\\ 0 & 1 & 0 & 0\\ -\sin\theta & 0 & \cos\theta & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$



Rotation about Axis Parallel to X

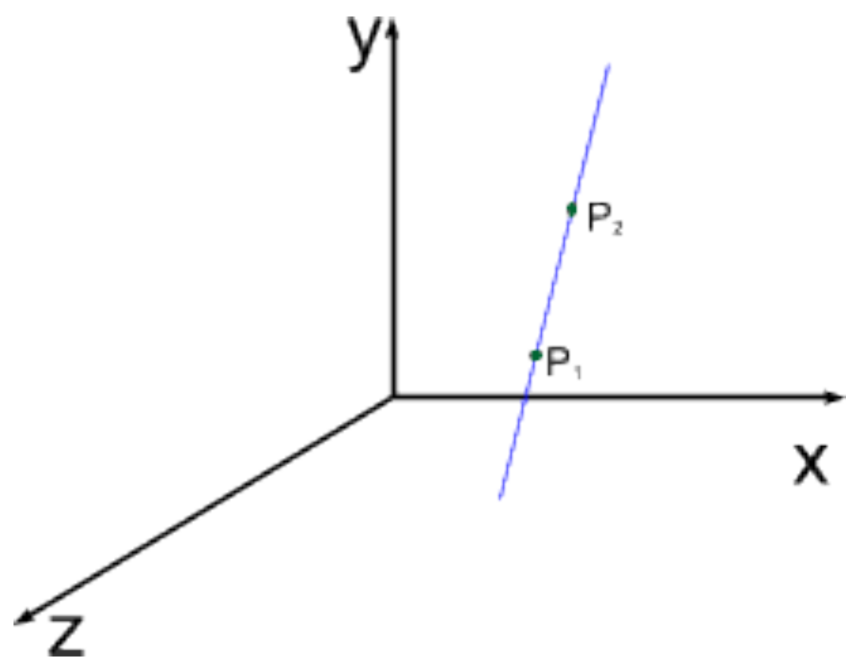
- Translate rotation axis to X
- Rotate
- Translate back





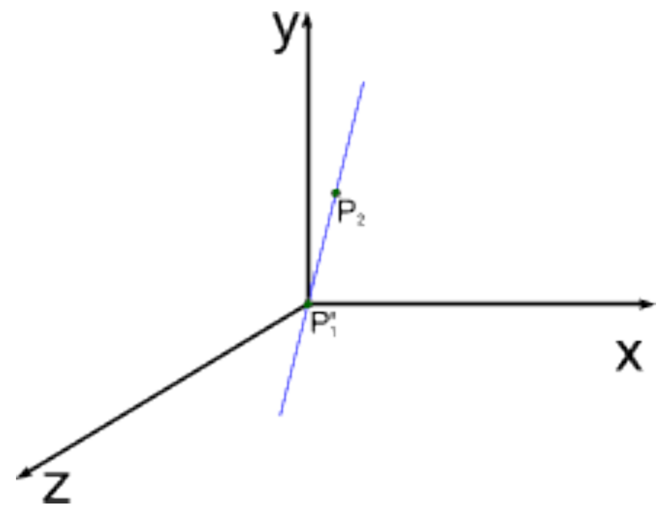
Rotation about Arbitrary Axis

- To rotate about axis through P_1P_2

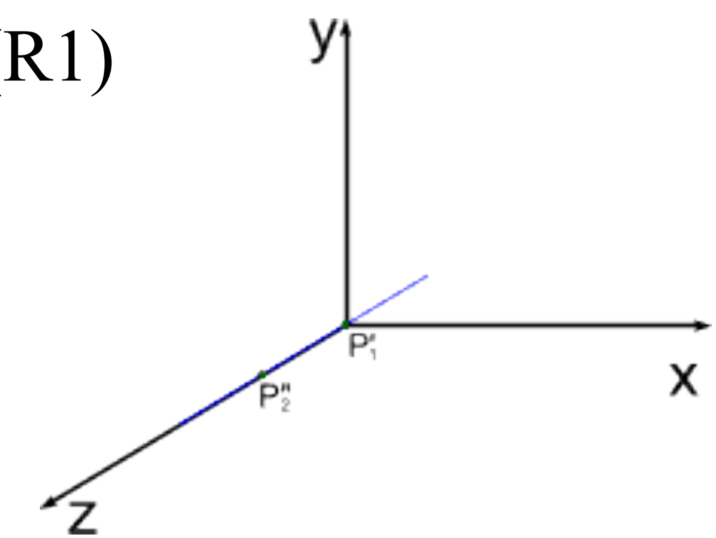


Rotation about Arbitrary Axis (2)

- Translate P_1 to origin (T)

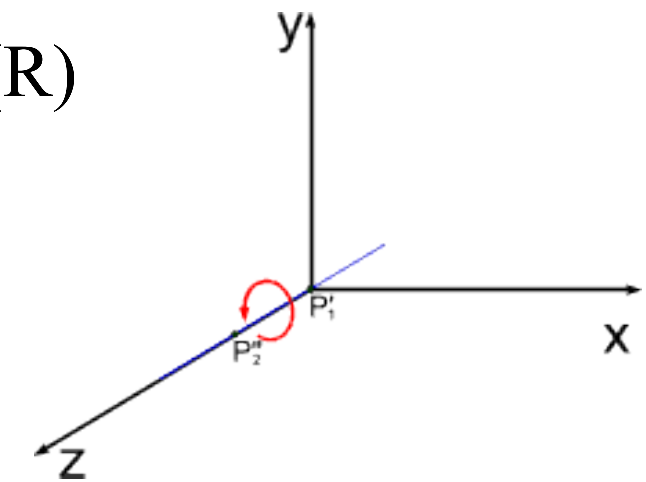


- Rotate until P_2 lies on z axis (R1)

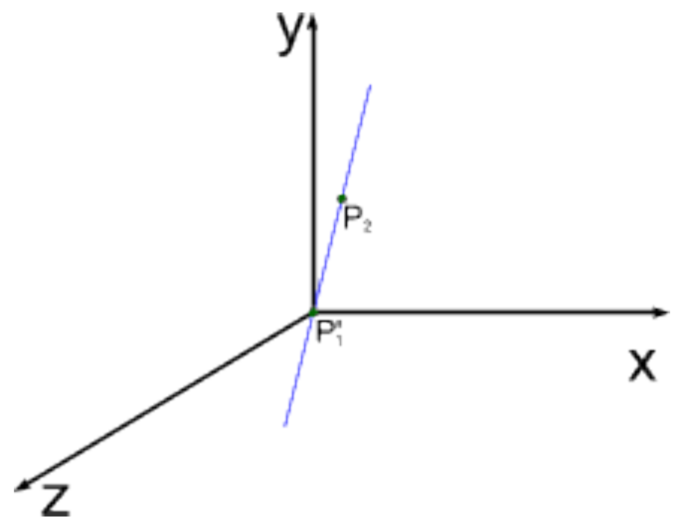


Rotation about Arbitrary Axis (3)

- Perform desired rotation (R)

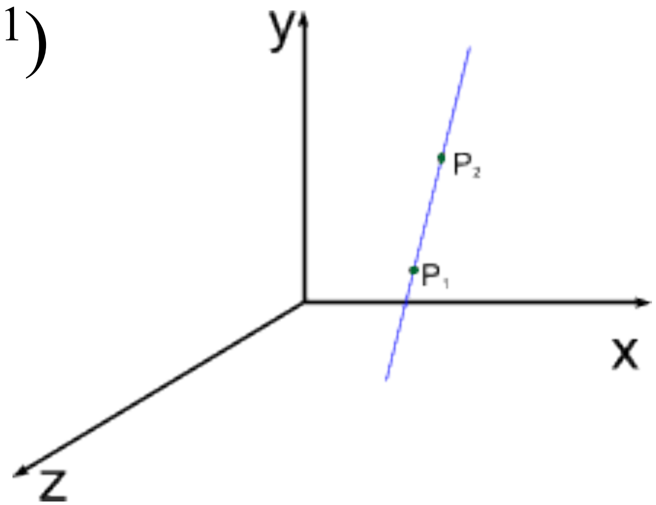


- Rotate axis back (R_1^{-1})



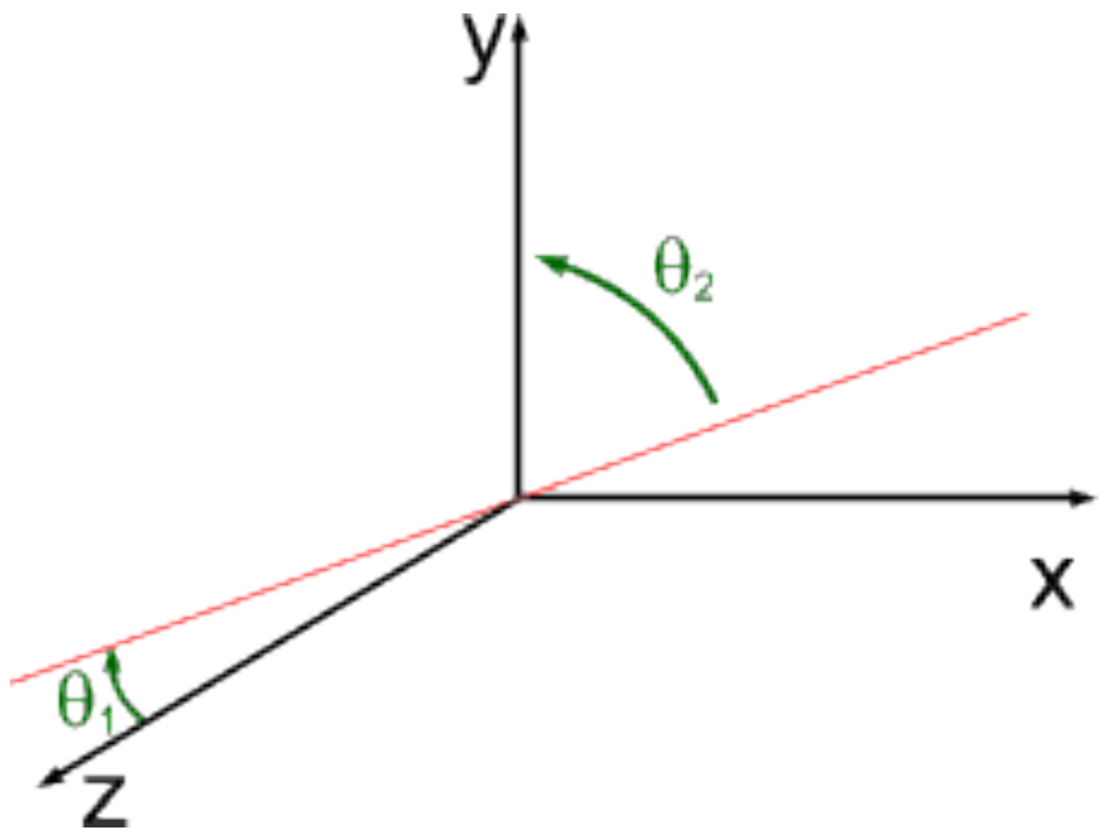
Rotation about Arbitrary Axis (4)

- Translate axis back (T^{-1})



Rotation Axis Specification

1. Point and two rotation angles

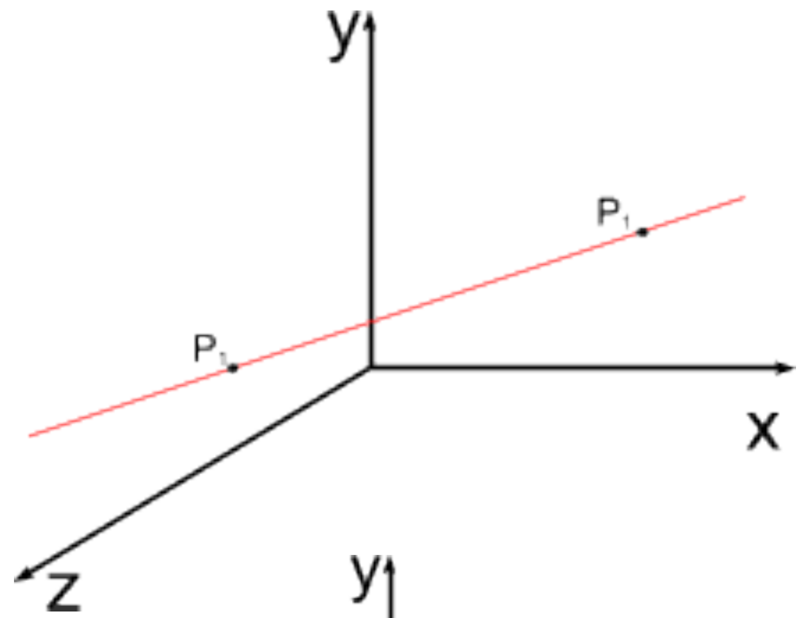


Rotation Axis Specification

2. Two points

$V = P_2 - P_1$

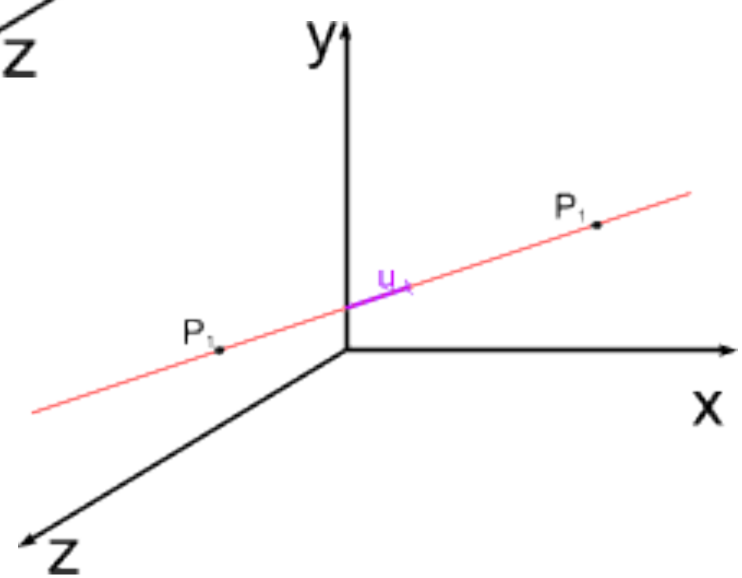
$U = V / |V| = (a,b,c)$



$a = (x_2 - x_1) / |V|$

$b = (y_2 - y_1) / |V|$

$c = (z_2 - z_1) / |V|$



Rotating Axis onto Coordinate Axis

1. Rotate about x-axis into xz plane

Same as rotation of yz projection onto z

u’ = (0,b,c)

$u_2 = (0,0,d)$

$\cos \alpha = (u' \bullet u_2)/(|u'||u_2|$

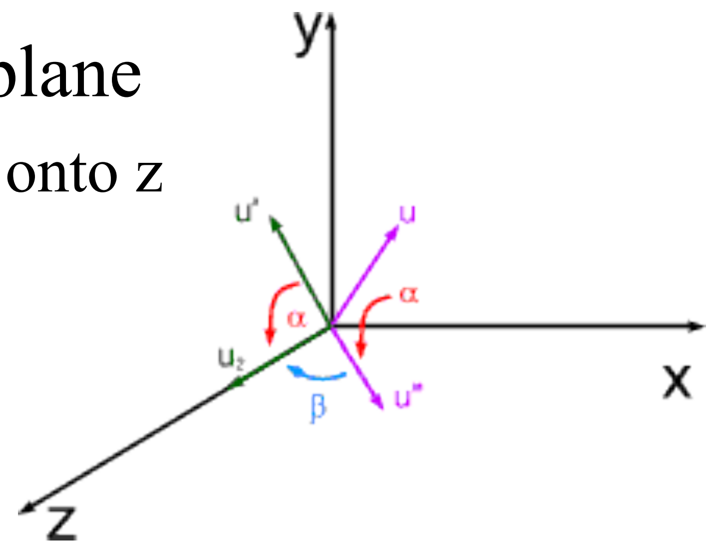
$= cd/dd$

$= c/d$

where $d = \text{sqrt}(b^2+c^2)$

$\sin \alpha = b/d$

u” = (a,0,d)



$$R_x(x)=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c/d & -b/d & 0 \\ 0 & b/d & c/d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotating Axis onto Coordinate Axis

1. Rotate about x-axis into xz plane
2. Rotate about y onto x axis

$$R_y(\alpha) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c/d & -b/d & 0 \\ a & b/d & c/d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_y(\beta) = \begin{bmatrix} d & 0 & -a & 0 \\ 0 & 1 & 0 & 0 \\ a & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- composite rotation $R_1 = R_y(\beta)R_x(\alpha)$

Transforming Normals

- Transforming by same matrix as points doesn't necessarily work
- Can calculate correct matrix using relationship to tangent vector (which does transform correctly)
  - $N = (M^{-1})^T$

Coordinate Transformations

- Can move coordinate frame, rather than points
  - $\mathbf{p} + u\mathbf{u} + v\mathbf{v} + w\mathbf{w}$
  - origin $\mathbf{p}$, basis vectors $\mathbf{u},\mathbf{v},\mathbf{w}$
- Frame-to-canonical conversion

$$\mathbf{p}_{\mathbf{xy}z} = \begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} & \mathbf{e} \\ 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{p}_{\mathbf{uv}w}$$

$$\mathbf{p}_{uvw} = \begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} & \mathbf{e} \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \mathbf{p}_{xyz}$$