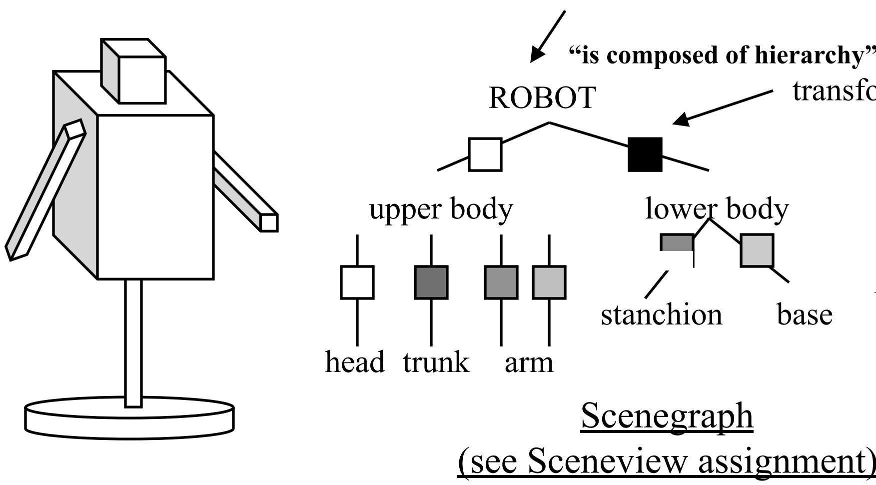
Geometric Transformations

Readings: Chapters 5-6

Why do we need geometric Transformations (T,R,S) in Graphics?

- Scene graph (DAG) construction from primitives

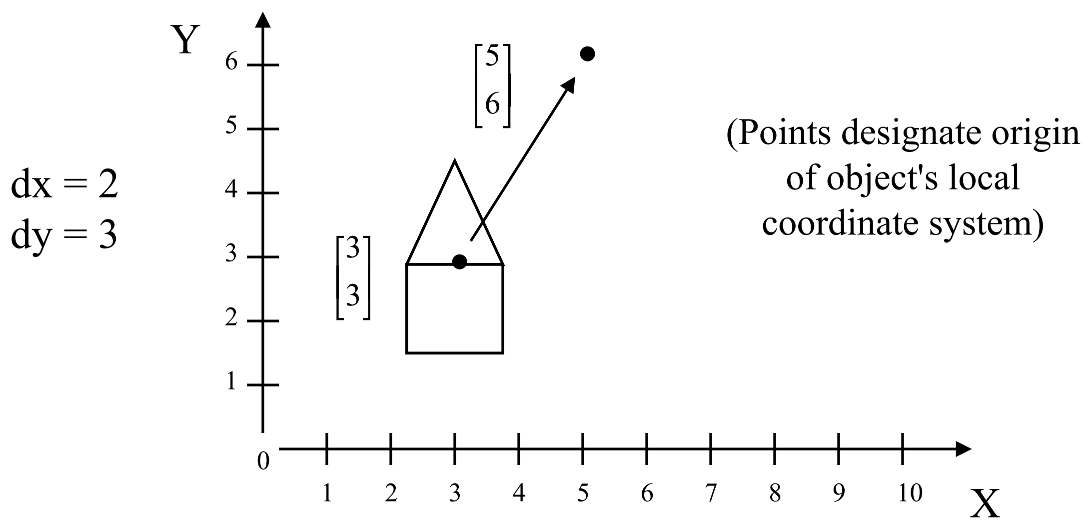


More on scenegraphs on Slide 18

- Object motion (this lecture)
- Camera motion (next lecture on viewing)

2D Transformations

2D Translation



- Component-wise addition of vectors

$$\boldsymbol{v}' = \boldsymbol{v} + \boldsymbol{t} \quad \text{where} \quad v = \begin{bmatrix} x \\ y \end{bmatrix}, \quad v' = \begin{bmatrix} x' \\ y' \end{bmatrix}, \quad t = \begin{bmatrix} dx \\ dy \end{bmatrix}$$

and $x' = x + dx$

$y' = y + dy$

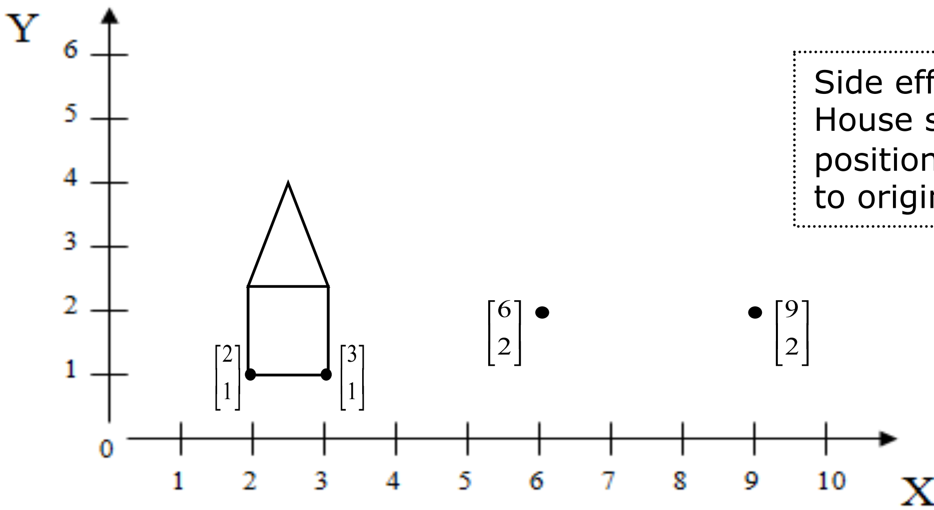
To move polygons: translate vertices (vectors) and redraw lines between them

- Preserves lengths (isometric)
- Preserves angles (conformal)
- Translation is thus "rigid-body"

2D Scaling

$s_x = 3$

$s_y = 2$



Side effect: House shifts position relative to origin

- Component-wise scalar multiplication of vectors

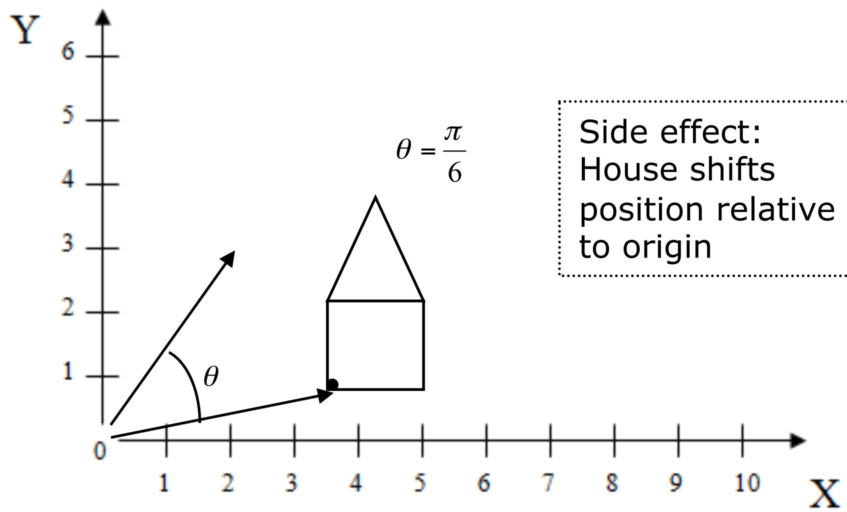
$$\boldsymbol{v'} = \boldsymbol{Sv} \quad \text{where} \quad v = \begin{bmatrix} x \\ y \end{bmatrix}, \quad v' = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

and $$S = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \qquad \begin{aligned} x' &= s_x x \\ y' &= s_y y \end{aligned}$$

- Does not preserve lengths
- Does not preserve angles (except when scaling is uniform)

2D Rotation

Rotate by θ about the origin



Side effect: House shifts position relative to origin

$\boldsymbol{v'} = \boldsymbol{R_\theta}\, \boldsymbol{v}$ where $v = \begin{bmatrix} x \\ y \end{bmatrix}, \quad v' = \begin{bmatrix} x' \\ y' \end{bmatrix}$

and $x' = x \cos\Theta - y \sin\Theta$

$y' = x \sin\Theta + y \cos\Theta$

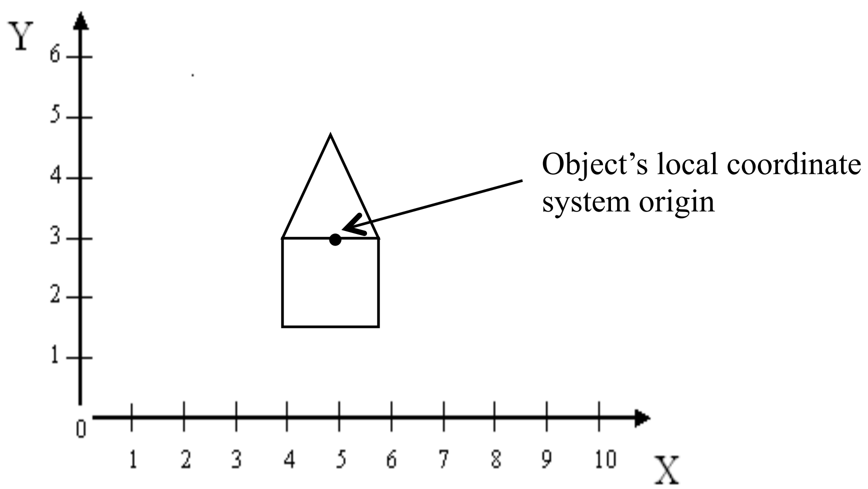
$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

A rotation by 0 angle, i.e. no rotation at all, gives us the identity matrix

- Preserves lengths in objects, and angles between parts of objects
- Rotation is rigid-body

2D Rotation and Scale are Relative to Origin

- Suppose object is not centered at origin and we want to scale and rotate it.
- Solution: move to the origin, scale and/or rotate *in its local coordinate system*, then move it back.



- This sequence suggests the need to compose successive transformations…

Homogenous Coordinates

- Translation, scaling and rotation are expressed as:

  translation: $v' = v + t$

  scale: $v' = Sv$

  rotation: $v' = Rv$

- Composition is difficult to express
  - translation is not expressed as a matrix multiplication
- Homogeneous coordinates allows expression of all three transformations as 3x3 matrices for easy composition

$$P_{2d}(x, y) \rightarrow P_h(wx, wy, w), \quad w \neq 0$$

$$P_h(x', y', w), \quad w \neq 0$$

$$P_{2d}(x, y) = P_{2d}\left(\frac{x'}{w}, \frac{y'}{w}\right)$$

- *w* is 1 for affine transformations in graphics
- Note: $\mathrm{p} = (\mathrm{x}, \mathrm{y})$ becomes $\mathrm{p} = (\mathrm{x}, \mathrm{y}, 1)$

  This conversion does not transform *p*. It is only changing notation to show it can be viewed as a point on *w = 1* hyperplane

2D Homogeneous Coordinate Transformations

- For points written in homogeneous coordinates,

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

translation, scaling and rotation relative to the origin are expressed homogeneously as:

dy)v

$$T_{(d_x, d_y)} = \begin{bmatrix} 1 & 0 & d_x \\ 0 & 1 & d_y \\ 0 & 0 & 1 \end{bmatrix} \qquad v' = T_{(d_x, d_y)} v$$

s_y)v

$$S_{(s_x, s_y)} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad v' = S_{(s_x, s_y)} v$$

(ϕ)v

$$R_{(\phi)} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad v' = R_{(\phi)} v$$

Examples

- Translate [1,3] by [7,9]

$$\begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 9 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 12 \\ 1 \end{bmatrix}$$

- Scale [2,3] by 5 in the X direction and 10 in the Y direction

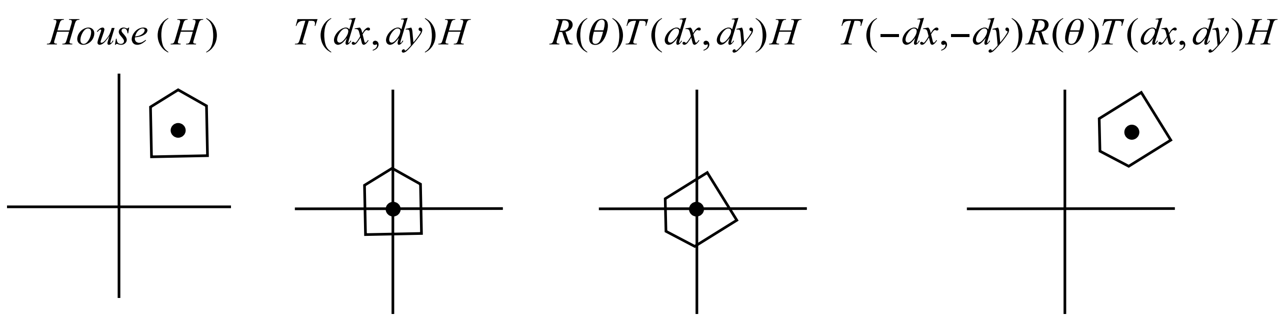
$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 30 \\ 1 \end{bmatrix}$$

- Rotate [2,2] by 90° (π/2)

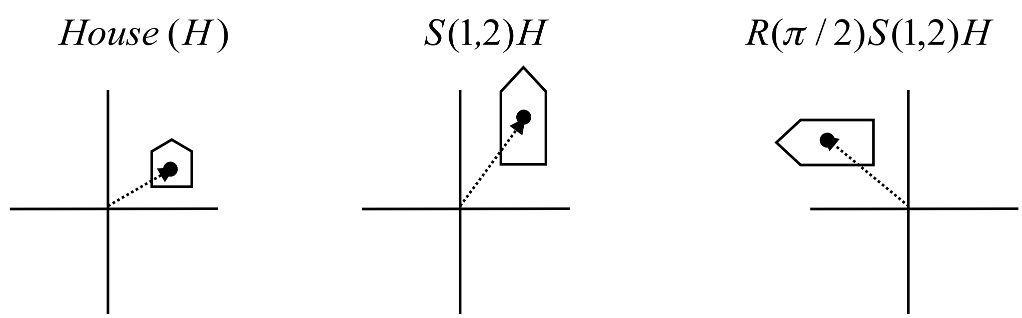
$$\begin{bmatrix} \cos(\pi/2) & -\sin(\pi/2) & 0 \\ \sin(\pi/2) & \cos(\pi/2) & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$$

Matrix Compositions: Using Translation

- Avoiding unwanted translation when scaling or rotating an object not centered at origin:
  - translate object to origin, perform scale or rotate, translate back.



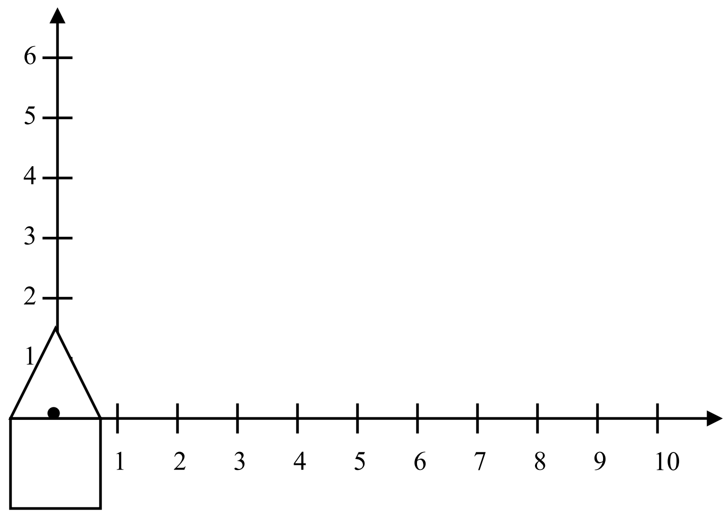
- How would you scale the house by 2 in “its” y and rotate it through 90° ?



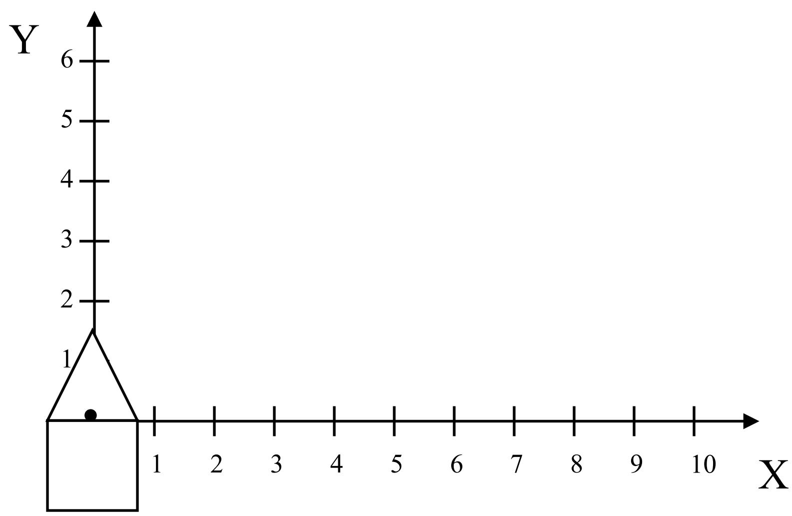
- Remember: matrix multiplication is <u>not</u> commutative! Hence order matters! (Refer to class website: resources/ SIGGRAPH2001_courseNotes_source/ transformation for camera transformation)

Matrix Multiplication is NOT Commutative

Translate by x=6, y=0 then rotate by 45°



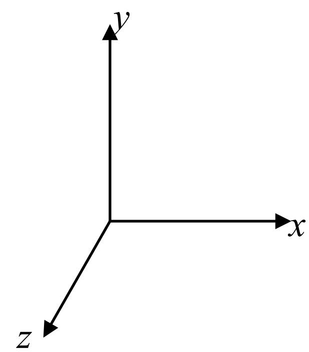
Rotate by 45° then translate by x=6, y=0



3D Transformations

3D Basic Transformations (1/2)

(right-handed coordinate system)



- Translation
$$\begin{bmatrix} 1 & 0 & 0 & dx \\ 0 & 1 & 0 & dy \\ 0 & 0 & 1 & dz \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Scaling
$$\begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3D Basic Transformations (2/2)

(right-handed coordinate system)

- Rotation about X-axis

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Rotation about Y-axis

$$\begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Rotation about Z-axis

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

More transformation matrices

Skew/Shear/Translate (1/2)

- "Skew" a scene to the side:

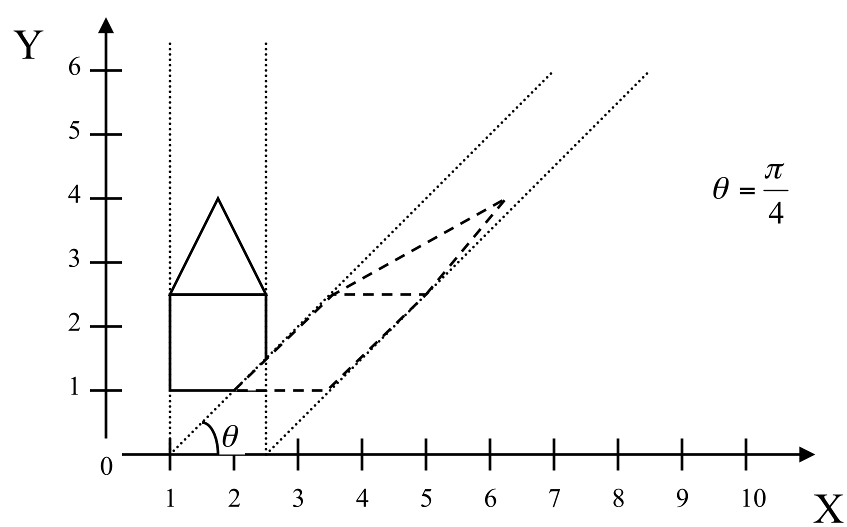
$$Skew_\theta = \begin{bmatrix} 1 & \frac{1}{\tan\theta} \\ 0 & 1 \end{bmatrix}$$

2D non-homogeneous

$$Skew_\theta = \begin{bmatrix} 1 & \frac{1}{\tan\theta} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2D homogeneous

- Squares become parallelograms - x coordinates skew to right, y coordinates stay same
- 90° between axes becomes θ
- Like pushing top of deck of cards to the side – each card shifts relative to the one below
- Notice that the base of the house (at y=1) remains horizontal, but shifts to the right.



NB: A skew of 0 angle, i.e. no skew at all, gives us the identity matrix, as it should

Scene graph manipulation

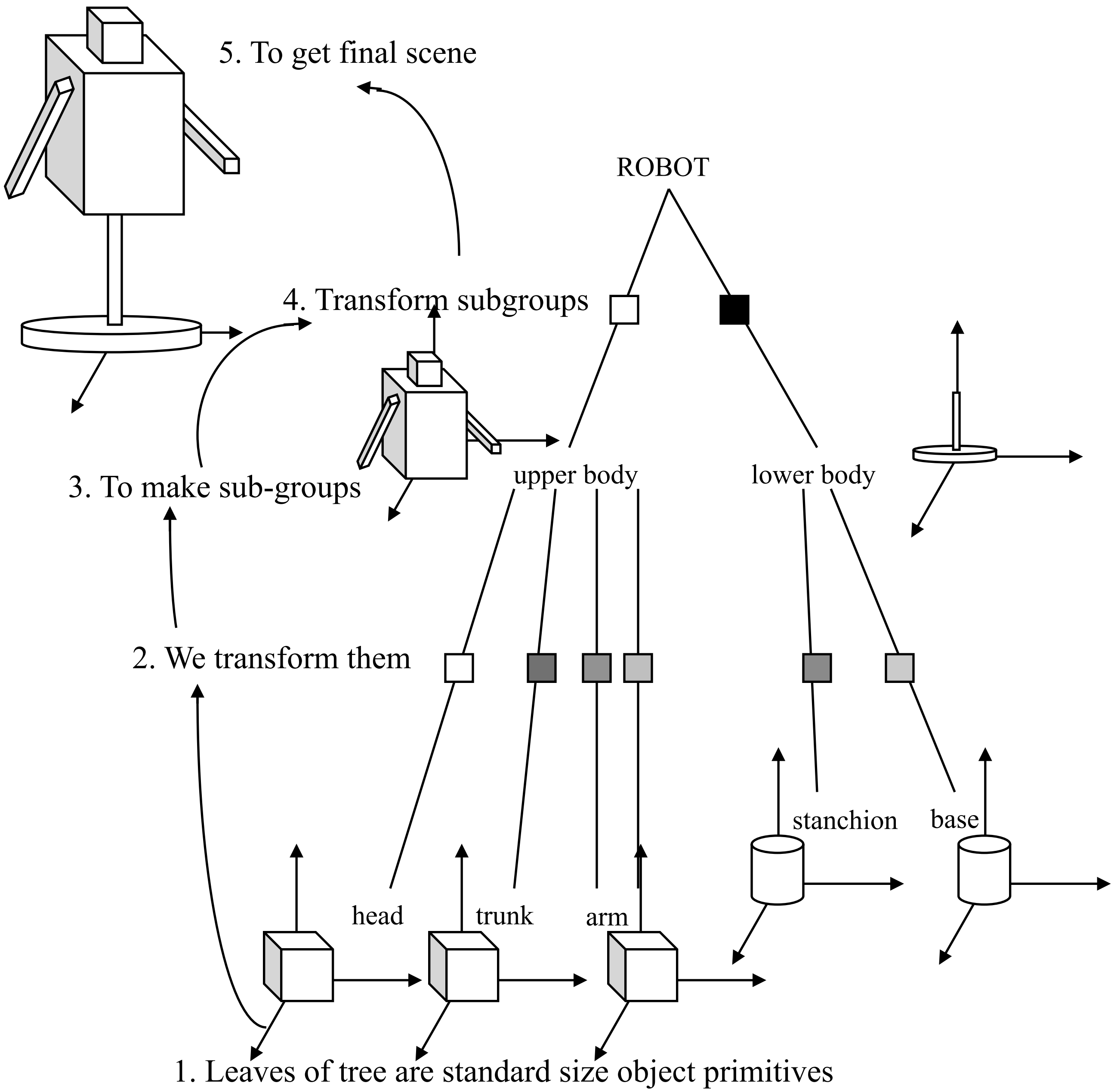
Transformations in Scene Graphs (1/3)

- 3D scenes are often stored in a *scene graph:*
  - Open Scene Graph
  - Sun’s Java3D™
  - X3D ™ (VRML ™ was a precursor to X3D)

- Typical scene graph format:
  - **objects** (cubes, sphere, cone, polyhedra etc.)
    - stored as nodes (default: unit size at origin)
  - **attributes** (color, texture map, etc.) and **transformations** are also nodes in scene graph (labeled edges on slide 2 are an abstraction)

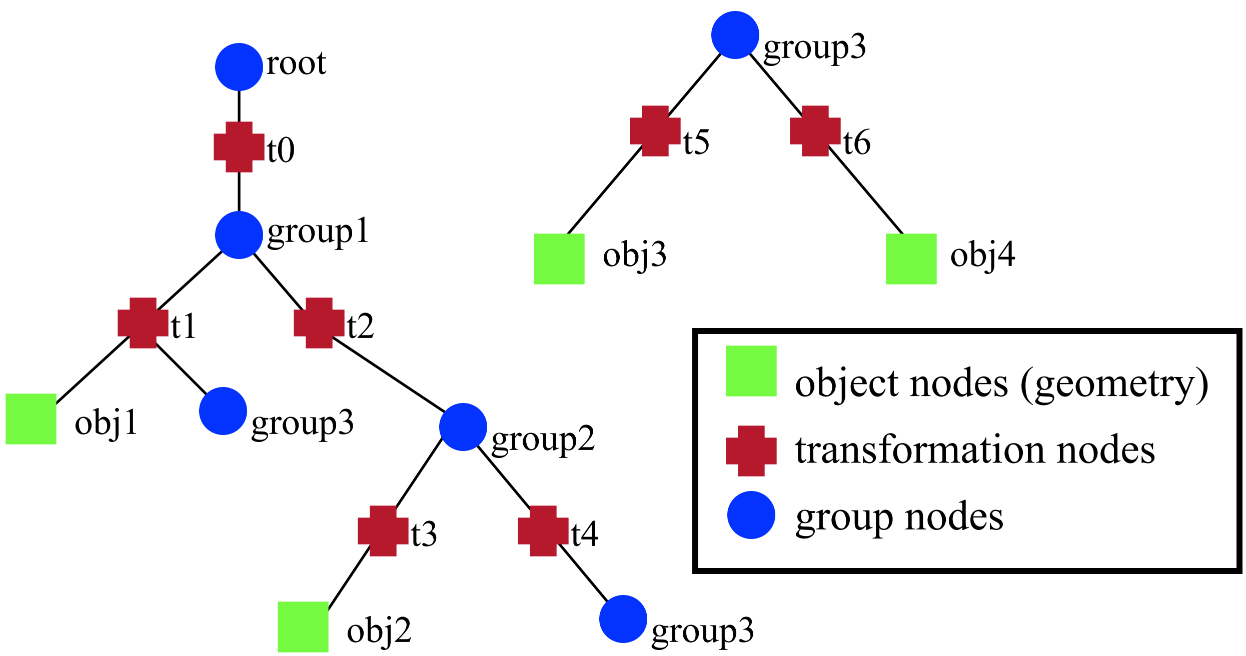
Transformations in Scene Graphs (2/3)

Closer look at Scenegraph from slide 2 …



Transformations in Scene Graphs (3/3)

- Transformations affect all child nodes
- Sub-trees can be reused, called group nodes
  - instances of a group can have different transformations applied to them (e.g. group3 is used twice– once under t1 and once under t4)
  - must be defined before use

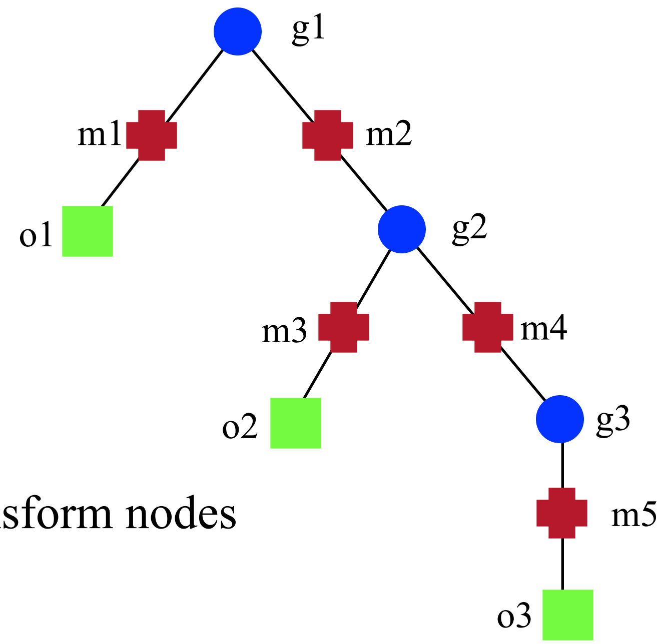


Composing Transformations in a Scene Graph (1/2)

- Transformation nodes contain at least a matrix that handles the transformation;
  - may also contain individual transformation parameters
- To determine final composite transformation matrix (CTM) for object node:
  - compose all parent transformations during prefix graph traversal
  - exact detail of how this is done varies from package to package, so be careful

Composing Transformations in a Scene Graph (2/2)

- Example:



g: group nodes

m: matrices of transform nodes

o: object nodes

- for o1, CTM = m1
- for o2, CTM = m2* m3
- for o3, CTM = m2* m4* m5

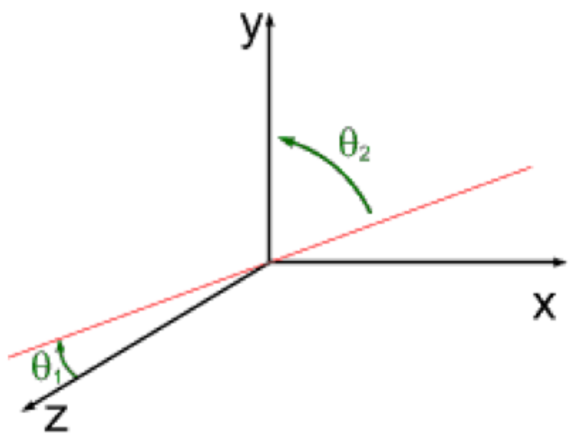
- for a vertex v in o3, position in the world (root) coordinate system is:
CTM v = (m2*m4*m5)v

Excercies

Rotation axis specification

(right-handed coordinate system)

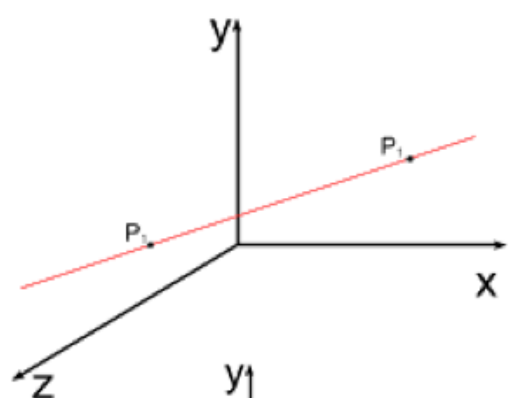
1. Point and two rotation angles



2. Two points

$V = P_2 - P_1$

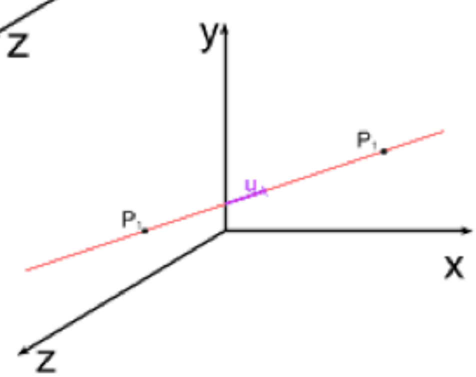
$U = V / |V| = (a,b,c)$



$a = (x_2 - x_1) / |V|$

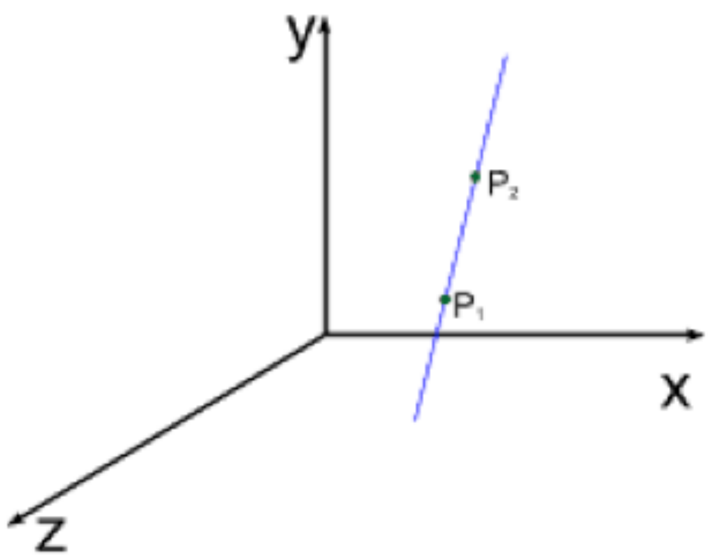
$b = (y_2 - y_1) / |V|$

$c = (z_2 - z_1) / |V|$



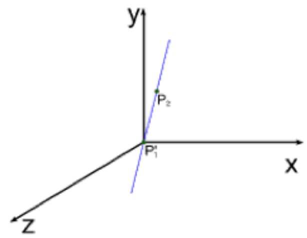
Rotation about arbitrary axis

- To rotate about axis through P_1P_2

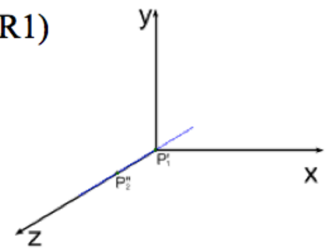


Rotation about arbitrary axis

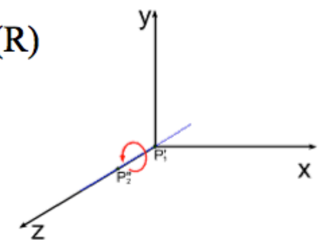
- Translate P_1 to origin (T)



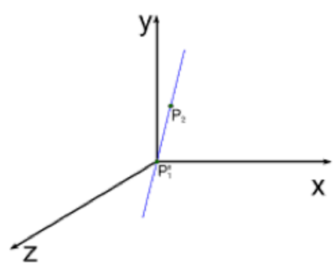
- Rotate until P_2 lies on z axis (R1)



- Perform desired rotation (R)



- Rotate axis back (R_1^{-1})



- Translate axis back (T^{-1})