T minus 6 classes

- Quiz on Probability **next** class
  - Know material on the slides we covered
- Homework 8 will be due in one week
- Slight changes in schedule
  - Should give more time on HW8 and HW9

CMSC 203: Lecture 21

More Probability

Bayes' Theorem

- Suppose that *E* and *F* are events from a sample space *S* such that $p(E) \neq 0 \text{ and } p(F) \neq 0$

$$p(H|E) = \frac{p(E|H)p(H)}{p(E)}$$

$$p(H|E) = \frac{p(E|H)p(H)}{p(E|H)p(H) + p(E|\bar{H})p(\bar{H})}$$

Bayes' Theorem Terms

$$p(H|E) = \frac{p(E|H)p(H)}{p(E)}$$

$$p(H|E) = \frac{p(E|H)p(H)}{p(E|H)p(H) + p(E|\bar{H})p(\bar{H})}$$

- **H**: Hypothesis
- **E**: Evidence
- **p(H)**: "Prior" probability of H
- **p(H|E)**: "Posterior" probability
- **p(E|H)**: "Likelihood"
- **p(E)**: Normalizing constant

Bayes' Theorem Application

- There is a rare disease that 1 in 100,000 people has. There is a test that is correct 99.0% of the time when the person has the disease, and 99.5% correct when testing a person who does not have the disease. Can we find:
  - probability a person who tests positive has disease?
  - probability a person who tests negative doesn't?

	Disease :(	**No disease! :)**
Positive Test	**CORRECT**	False positive
Negative Test	False negative	**CORRECT**

Random Variables

- A **function** from *sample space* of an *experiment* to the set of **real numbers**
  - A random variable assigns a real number to each possible outcome
  - Not actually a variable; not actually random
    - Your book hates this, too

Example of Random Variable

- Let $X(t)$ be the random variable that equals the number of heads that appear when t is the outcome

$$X(HHH) = 3$$
$$X(HHT) = X(HTH) = X(THH) = 2$$
$$X(TTH) = X(THT) = X(HTT) = 1$$
$$X(TTT) = 0$$

Distribution of Random Variable

- The **distribution** of a random variable *X* on sample space *S* is the set of pairs $(r, p(X = r))$ for all $r \in X(S)$ where $p(X = r)$ is the probability *X* takes the value *r*
- *Example* : Taking 3 coin flips from the previous example

  $P(X = 3) = 1/8$ $P(X = 2) = 3/8$ $P(X = 1) = 3/8$
  $P(X = 0) = 1/8$

- Therefore, the distribution of $X(t)$ is the set of pairs:
  $\{(3, 1/8), (2, 3/8), (1, 3/8), (1, 1/8)\}$

Examples of Random Variables

- Sum of numbers when dice is rolled
- Amount of rain (or snow) that falls on a particular day
- How many goats you can win in Monty Hall problem
- All probability distributions

Expected Value

- **Expected value**: The average value of a random variable when an experiment is performed many times
  - Number of heads expected to show up?
  - Expected number of comparisons to find element in a list using a linear search?
  - Expected value of playing the lottery / poker / drilling giant holes into the Earth to find oil
- Also called the *expectation* or *mean* of a random variable

Expected Value Formula

- Expected value of random variable *X* on sample space *S*

$$E(X) = \sum_{s \in S} p(s)X(s)$$

- *Example:* Let *X* be the number that comes up when a die is rolled. What is the expected value of *X*?
- The final exam of a discrete mathematics course consists of **50** true/false questions, each worth **two points**, and **25** multiple-choice questions, each worth **4 points**. The probability that a student answers a T/F question correctly is **.9** and the chance they answer a multiple choice question correctly is **.8**. What is their expected score?

St. Petersburg Paradox

- There's a new gambling game at the casino:
  - Single player game, consisting of 1 fair coin
  - The pot starts at $1, and the coin is flipped:
    - If heads, the pot is doubled
    - If tails, the game ends and you win the pot
  - **tl;dr** - you win 2^n dollars if heads comes on $n\mathrm{th}$ flip
- What is the expected value of this game? How much would you pay to play this game?

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Relations

What is Relations?

- **Relation**: A strcture that represents the relationships between elements of sets
- Subset of the Cartesian product of the sets
- Examples:
  - Pairs of cities linked by airline flights?
  - Phases of a project in a viable order?
  - Storing information in a database?

Expressing Relations

- Most direct way to express relationship between elements in two sets is *ordered pairs*
- Sets of ordered pairs are **binary relations**
  - Binary relation from A to B is a subset of A x B
  - $aRb \text{ denotes } (a, b) \in R$
- *Example* : Let A be the set of students at UMBC and B be the set of courses. Let *R* be the relation that consist of pairs (*a, b*) where *a* is a student enrolled in course *b*.

Functions as Relations

- Recall function f from A to B maps elements exactly one element in B to every element in A
- Graph of f is a set of ordered pairs (a, b) where $b = f(a)$
- Graph of f is a subset of A x B, so it is a relation
- Relations are a generalizations of graphs of functions

Relations on a Set

- A **relation on a set A** is a relation from A to A
- *Example* s:
  - A is the set {1, 2, 3, 4}. Which ordered pairs are in the relation *R* = {(a, b) | a divides b}?
  - Which of these relations contain the pair (1, 2):
    - $R_1 = \{(a, b) | a \leq b\}$
    - $R_2 = \{(a, b) | a = b + 1\}$
    - $R_3 = \{(a, b) | a + b \leq 3\}$

Properties of Relations

- **Reflexive**
  - $\forall a((a,a) \in R)$
- **Symmetric**
  - $\forall a \forall b((a,b) \in R \rightarrow (b,a) \in R)$
- **Antisymmetric**
  - $\forall a \forall b(((a,b) \in R \land (b,a) \in R) \rightarrow (a = b))$
- **Transitive**
  - $\forall a \forall b \forall c(((a,b) \in R \land (b,c) \in R) \rightarrow (a,c) \in R)$

Combining Relations

- Two relations from A to B can be combined in any way two sets can be combined
  - Due to relations being subsets of A x B
- We can perform operations such as
  - $R_1 \cup R_2$
  - $R_1 \cap R_2$
  - $R_1 - R_2$
  - $R_2 - R_1$
  - $R_1 \oplus R_2$