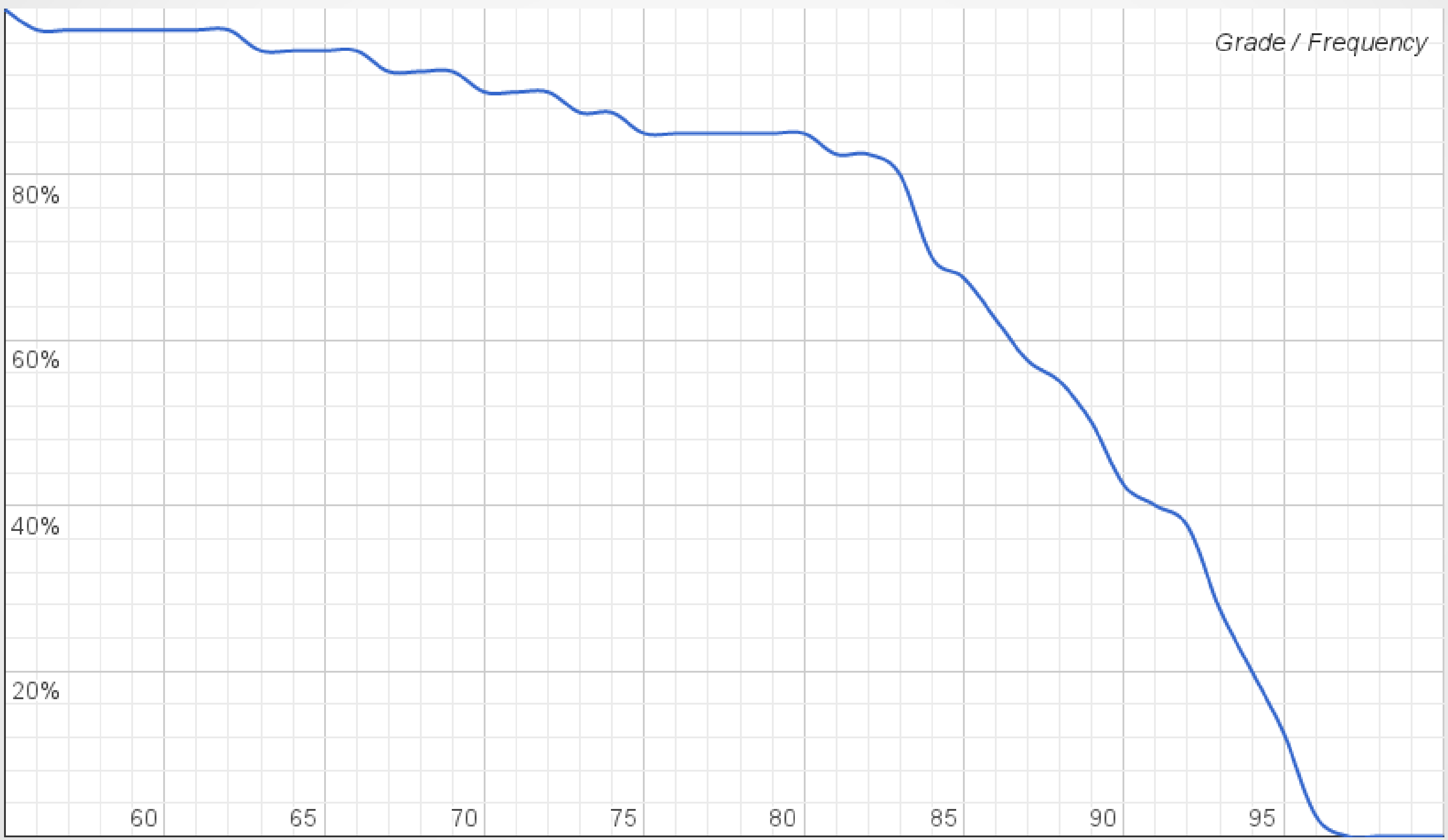
T minus 7 classes

- Quiz on Probability next class
  - Know material on the slides we covered
- Homework will be released in a few hours
  - Will be based on how far we get today
  - Due in one week (11/19)
- Exams are being graded (along with Homework 6 and 7)

Current Grade Status



CMSC 203: Lecture 20

Probably Probability

(I hope the projector works today)

General Probabilities

- For sample space *S* with countable outcomes, probability *p(s)* for each outcome *s* meets conditions:
  1) $0 \leq p(s) \leq 1 \text{ for each } s \in S$
  2) $\sum_{s \in S} p(s) = 1$
- Function *p* is the **probability distribution**
- *p(s)* should equal limit of the times *s* occurs divided by number of times experiment is performed (as experiment count grows without bound)

Probability Distributions

- If *S* is a set with *n* elements, a **uniform distribution** assigns probability of 1/*n* for each element of *S*
- $$p(E) = \sum_{s \in E} p(s)$$
- Selecting element from sample space with uniform distribution is selecting an element **at random**
- *Example* : What is probability of rolling an odd number on a dice if the dice is loaded so 3 comes up twice as often as each other number?

Conditional Probability

- **Conditional probability**: Probability E will occur given F, where E and F are events with p(F) > 0

$$p(E|F) = \frac{p(E \cap F)}{p(F)}$$

- Examples:
  - Bit string of length 4 is generated at random. What is the probability it contains two consecutive 0s given that the first bit is 0?
  - What is the probability a family will have two boys, given they already have one boy?

Independence

- When two events are **independent**, the occurrence of one of the events gives does not affect the other
- Two events are independent $\iff p(E \cap F) = p(E)P(F)$
- Example: *E* is an the event that a randomly generated bit string of length 4 begins with a 1, and *F* is the event that this bit string contains an even number of 1s. Are *E* and *F* independent?

The Birthday Problem

- What is the minimum number of people who need to be in a room so hat the probability that at least two of them have the same birthday is greater than 50%?

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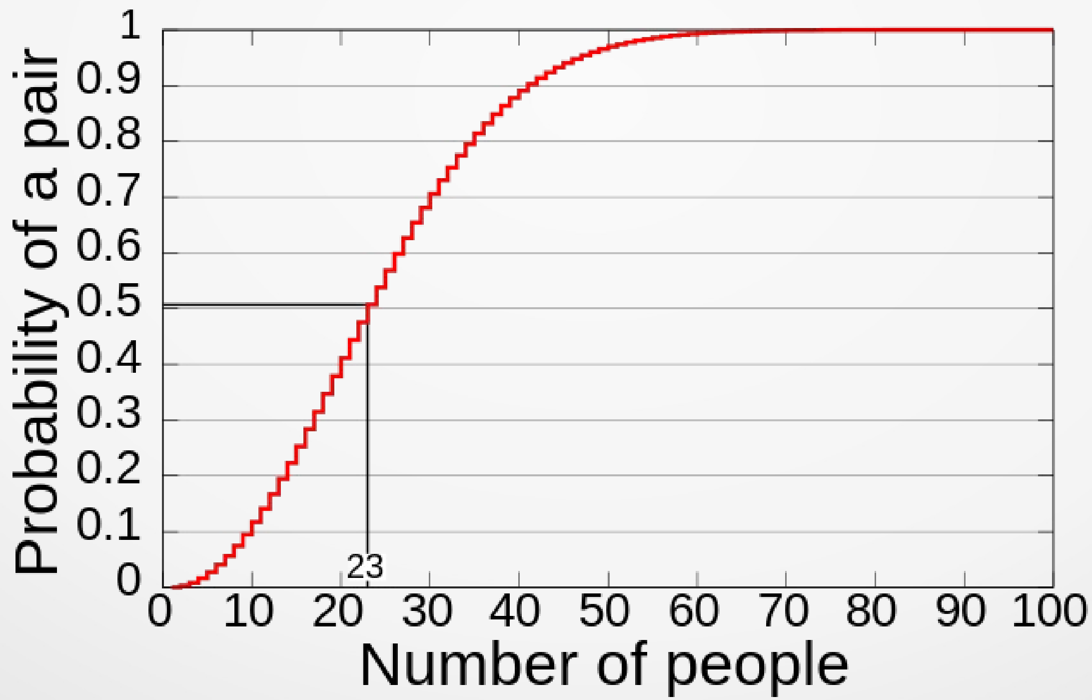

Some CS Applications

- Hash Tables
- Monte Carlo
  - Primality Testing
  - Monte Carlo Search Trees

Bayes' Theorem

- Suppose that *E* and *F* are events from a sample space *S* such that $p(E) \neq 0 \text{ and } p(F) \neq 0$

$$p(H|E) = \frac{p(E|H)p(H)}{p(E)}$$

$$p(H|E) = \frac{p(E|H)p(H)}{p(E|H)p(H) + p(E|\bar{H})p(\bar{H})}$$

Bayes' Theorem Terms

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- **H**: Hypothesis
- **E**: Evidence
- **p(H)**: “Prior” probability of H
- **p(H|E)**: “Posterior” probability
- **p(E|H)**: “Likelihood”
- **p(E)**: Normalizing constant

Bayes' Theorem Application

- There is a rare disease that 1 in 100,000 people has. There is a test that is correct 99.0% of the time when the person has the disease, and 99.5% correct when testing a person who does not have the disease. Can we find:
  - probability a person who tests positive has disease?
  - probability a person who tests negative doesn't?

	Disease :(	**No disease! :)**
Positive Test	**CORRECT**	False positive
Negative Test	False negative	**CORRECT**

Random Variables

- A **function** from *sample space* of an *experiment* to the set of **real numbers**
  - A random variable assigns a real number to each possible outcome
  - Not actually a variable; not actually random
    - Your book hates this, too

Example of Random Variable

- Let $X(t)$ be the random variable that equals the number of heads that appear when t is the outcome

$$X(HHH) = 3$$
$$X(HHT) = X(HTH) = X(THH) = 2$$
$$X(TTH) = X(THT) = X(HTT) = 1$$
$$X(TTT) = 0$$

Distribution of Random Variable

- The **distribution** of a random variable *X* on sample space *S* is the set of pairs $(r, p(X = r))$ for all $r \in X(S)$ where $p(X = r)$ is the probability *X* takes the value *r*
- *Example* : Taking 3 coin flips from the previous example

$P(X = 3) = 1/8$ $P(X = 2) = 3/8$ $P(X = 1) = 3/8$
$P(X = 0) = 1/8$

- Therefore, the distribution of $X(t)$ is the set of pairs:
  $\{(3, 1/8), (2, 3/8), (1, 3/8), (1, 1/8)\}$

Examples of Random Variables

- Sum of numbers when dice is rolled
- Amount of rain (or snow) that falls on a particular day
- How many goats you can win in Monty Hall problem
- All probability distributions

Expected Value

- **Expected value**: The average value of a random variable when an experiment is performed many times
  - Number of heads expected to show up?
  - Expected number of comparisons to find element in a list using a linear search?
  - Expected value of playing the lottery / poker / drilling giant holes into the Earth to find oil
- Also called the *expectation* or *mean* of a random variable

Expected Value Formula

- Expected value of random variable *X* on sample space *S*

$$E(X) = \sum_{s \in S} p(s) X(s)$$

- *Example:* Let *X* be the number that comes up when a die is rolled. What is the expected value of *X*?
- The final exam of a discrete mathematics course consists of **50** true/false questions, each worth **two points**, and **25** multiple-choice questions, each worth **4 points**. The probability that a student answers a T/F question correctly is **.9** and the chance they answer a multiple choice question correctly is **.8**. What is their expected score?

St. Petersburg Paradox

- There's a new gambling game at the casino:
  - Single player game, consisting of 1 fair coin
  - The pot starts at $1, and the coin is flipped:
    - If heads, the pot is doubled
    - If tails, the game ends and you win the pot
  - **tl;dr** - you win 2^n dollars if heads comes on $n\mathrm{th}$ flip
- What is the expected value of this game? How much would you pay to play this game?