Reminders

- Today
  - Change of schedule: Focus on chapter 6.4 and 6.5
- Tomorrow
  - Homework 7 is due
  - Quiz on counting
  - Review for Exam #2
- Tuesday
  - Exam 2

Combination Practice

- How many ways can you select 6 crew members to go to Mars from 30 trained astronauts?
- How many bit strings of length n contain exactly r 1s?
- Suppose there are 9 faculty members in Math, and 11 in CS. How many ways can you select a committee that consists of 3 math faculty and 4 CS faculty?

Combination Practice

- How many ways can you select 6 crew members to go to Mars from 30 trained astronauts?
  - $\frac{30!}{6!24!} = \frac{30 * 29 * 28 * 27 * 26 * 25}{6 * 5 * 4 * 3 * 2 * 1} = 593,775$
- How many bit strings of length *n* contain exactly *r* 1s?
  - $C(n, r)$
- Suppose there are 9 faculty members in Math, and 11 in CS. How many ways can you select a committee that consists of 3 math faculty and 4 CS faculty?
  - $C(9, 3) * C(11, 4) = \frac{9!}{3!6!} * \frac{11!}{4!7!} = 84 * 330 = 27,720$

More on Mars

- The group of 30 consists of 22 men and 8 women. How many ways are there to select a crew of 6 that must have at least 2 men and at least 2 women?

More on Mars

- The group of 30 consists of 22 men and 8 women. How many ways are there to select a crew of 6 that must have at least 2 men and at least 2 women?
  - Product rule for different groups:
    - 4 men / 2 women
    - 3 men / 3 women
    - 2 men / 4 women
  - Sum rule on all three possible layouts

CMSC 203: Lecture 17

Counting **3**
Advanced Counting

Binomial Expressions

- A **binomial expression** is in the form of *x* + *y*
- We want to expand (x + y)3 (**binomial expansion**)
- $$\begin{aligned}(x+y)^3 &= (x+y)(x+y)(x+y) \\ &= (x^2+xy+y^2)(x+y) \\ &= x^3+x^2y+x^2y+xy^2+yx^2+xy^2+xy^2+y^3 \\ &= x^3+3x^2y+3xy^2+y^3\end{aligned}$$
- Pattern of $x^i y^j$ is easy, but we want to know coefficients

Binomial Coefficients

- Observe that we may find number of x^2y elements by counting how many ways to select 2 *x* and a *y*

$$
\begin{aligned}
(x+y)^3 &= (x+y)(x+y)(x+y) \\
&= \binom{3}{3}x^3 + \binom{3}{2}x^2y + \binom{3}{1}xy^2 + \binom{3}{0}y^3 \\
&= x^3 + 3x^2y + 3xy^2 + y^3
\end{aligned}
$$

- This is why we call them **binomial coefficients**

The Binomial Theorem

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i$$

$$= \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \ldots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n$$

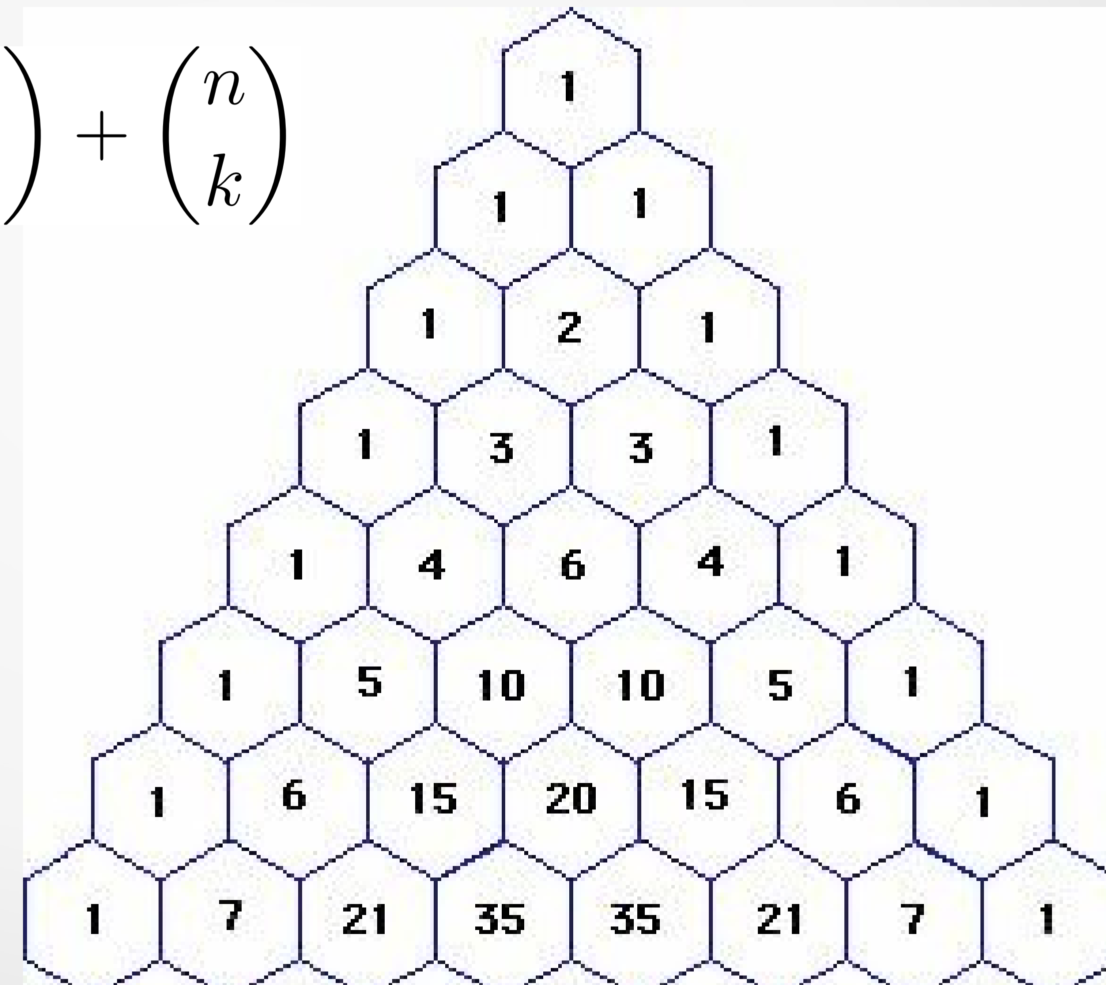
- We may use this to get coefficient for single term
- Examples:
  - Expansion of $(x+y)^4$?
  - Coefficient of $x^{12}y^{13}$ in the expansion of $(x+y)^{25}$?

Pascal's Identity and Triangle

- Pascal's Identity

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

- Pascal's Triangle
  - Find binomial coefficients with addition instead of multiplication



More Counting Problems

- Many types of counting problems:
  - Repetition / replication of objects
    - Selecting donut flavors
  - Indistinguishable elements
    - Arrange letters in *SUCCESS*
  - Distinguishable boxes
    - Dealing hands to 4 players

Permutations with Repetition

- P(n, r) with repetition is n^r
  - We know this from the product rule
- Example: How many strings of length *r* can be formed from the uppercase English letters (A-Z)?

Combinations with Repetition

- There are $C(n + r - 1, r)$ *r*-combinations from a set with *n* elements with repetition
  - This is a bit less intuitive
  - Think borders and stars

$$= \frac{(n + r - 1)!}{r!(n - 1)!}$$

- Example: How many ways are there to select 5 bills from a box containing an arbitrary number of \$1, \$2, \$5, \$10, \$20, \$50, and \$100 bills?

More Practice of Repetition

- How many ways are there to assign three jobs to five employees if each employee can be given more than one job?
- Suppose a bakery has four kinds of cookies. How many ways can six cookies be chosen?

Indistinguishable Objects

- Number of permutations of n objects, where there are n_1 indistinguishable elements of type 1, n_2 indistinguishable elements of type 2, ..., and n_k indistinguishable elements of type k is:

$$\frac{n!}{n_1!n_2!\ldots n_k!}$$

- Example: How many different strings can be made by reordering the letters of the word *SUCCESS*?

Distributing into Boxes

- **Distinguishable** may also be considered **labeled**
- Distinguishable objects are “different” from each other
  - Indistinguishable objects are “identical”
- Boxes may also be distinguishable or indistinguishable

Distinguishable Objects and Boxes

- Number of ways to distribute *n* distinguishable objects into *k* distinguishable boxes so that n_i objects are placed into box *i*, *i* = 1, 2, …, *k* equals:

$$\frac{n!}{n_1!n_2!\ldots n_k!}$$

- Example: How many ways are there to distribute hands of 5 cards to each of four players from the standard deck of 52 cards?

Indistinguishable objects; Distinguishable boxes

- Combination with repetition for *n* boxes and *r* objects:

$$C(n+r-1, r)$$

- Example: How many ways to distribute 50 index cards to a class of 40 students?

Otherwise...

- Distinguishable objects; Indistinguishable boxes
  - No closed form solutions
  - Example: Assigning employees to meeting rooms
- Indistinguishable objects and boxes
  - No closed form solutions
  - Can use partitions to show spread
  - Example: Packing identical books into boxes
- Both are outside scope of this class

Summary

- Permutations with repetition
- Combinations with repetition
- Permutations of indistinguishable objects
- Distributing:
  - Distinguishable objects and boxes
  - Indistinguishable objects; distinguishable boxes