Reminders

- Homework #3 is due Tuesday
- Homework #2 is due Tuesday if you used your late
- Tuesday is also a review day!
  - Bring questions you have on any material
- Thursday is the FIRST exam

CMSC 203: Lecture 9

Functions (since we didn't get to it yet)
Sequences, Series, and Cardinality

Functions

- Assigning elements of one set to another set (possibly the same)
- *Example*: Assigning set of students to set of grades
- This assignment is a **function** (also called **mapping** or **transformation**)
- Functions are extremely important to computer science
  - Definition of discrete structures
  - Runtime analysis
  - Calculating values (input → output)
  - Understanding recursive functions

Defining Functions

- Function f from A to B is assignment of **exactly one** element of B to **each** element of A
- Written as $f(x) = b$
  - Can be done as $f(x) = x + 1$
  - Can also be done by f(Apple) = Tree
- *b* is a unique element, assigned by f to element *a*
- If f is a function from A to B, write "f: A → B"

Definitions

If f is a function from *A* to *B*, and f(a) = b

- **Domain**: *A* is the domain of f
- **Codomain**: *B* is the codomain of f
- **Image**: *b* is the image of *a*
- **Preimage**: *a* is the preimage of *b*
- **Range**: Set of all images of elements of *A*
  - *Note:* This is **not** to be confused with codomain
- **Maps**: f maps *A* to *B*

Special Function Types

- **One-to-One / Injunctive**: f(a) = f(b) implies that a = b for all a and b in the domain
  - I.e., each image only has one preimage

$\forall a \forall b(f(a) = f(b) \rightarrow a = b)$ or $\forall a \forall b(a \neq b \rightarrow f(a) \neq f(b))$

- **Onto / Surjective**: Every element b ∊ B has element a ∊ A with f(a) = b
  - I.e., every element in the codomain is an image
  - $\forall y \exists x(f(x) = y)$
- **One-to-One correspondence / Bijective**: Function f is both, one-to-one and onto

Cardinality of Sets

- Two sets have the same **cardinality** if there is a one-to-one correspondence (bijunction) between the sets
- If there is a one-to-one function from A to B, $|A| \leq |B|$
- If a set is ininite, than $|S|$ is not a number
- If an infinite set has same cardinality as **Z+** then it is *countable*, otherwise it is *uncountable*

Function Operations

- **Inverse Function**
  - Denoted as f^{-1}
  - If f is a bijection, then $f^{-1}(b) = a$ whenever $f(a) = b$
  - i.e., f^{-1} assigns to b the element a such that $f(a) = b$
- **Composition**
  - Let f and g be functions from B to C and A to B (respectively)
  - $f(g(a)) = (f \circ g)(a)$ is the composition of f and g
  - Range of g must be a subset of the domain of f

Sequences

- **Sequences** are **lists** of **ordered** elements
  - Now order matters
- Can be finite or infinite
  - 1, 2, 3, 5, 8
  - 1, 3, 9, 27, 81, …, 3^n, …
- Formally, it is a function from **N** to some set **S**
  - a_n is the *n*th **term** of the sequence

Important Sequences

- **Geometric Sequence:** $f(x) = ar^x$, a ∈ **R**, r ∈ **R**
  - Initial term *a* and common ratio *r*
  - $c_n = 2 * 5^n$
    - 2, 10, 50, 250, 1250, …
- **Arithmetic Sequence:** $f(x) = dx + a$, d ∈ **R**, a ∈ **R**
  - Initial term *a* and common difference *d*
  - $d_n = -1 + 4n$
    - -1, 3, 7, 11, …

Recurrence Relations

- **Fibonacci sequence**: $f_n = f_{n-1} + f_{n-2}$, $f_0 = 0$, $f_1 = 1$
  - 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ...
- Defined using previous terms: **recurrence relations**
  - Can solve using *iteration*
  - Start at intial conditions, and work to a_n (**forward substitution**)
- Solve to express as a **closed formula** (in terms of *n*)
  - $a_n = a_{n-1} + 3$, $a_0 = 2$ can be expressed as $a_n = 2+3n$

Summations

- A **summation** of a sequence is the sume of all terms in series from a_m to a_n
- Written in the form: $\sum_{i=m}^{n} a_i$ or $\sum_{i=0}^{\infty} a_i = \sum_{a \in S} a$
- i is the **index of summation**, *m* is the **lower limit**, and n is the **upper limit**
- Normal arithmetic operations apply, such as sum of two summations

Useful Closed Forms

$$\sum_{k=0}^{n} a = (n+1)a$$

$$\sum_{k=0}^{n} ar^k = \frac{ar^{n+1} - a}{r-1}, (r \neq 0, r \neq 1)$$

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

$$\sum_{k=0}^{\infty} x^k, |x| < 1 = \frac{1}{1-x}$$

$$\sum_{k=1}^{\infty} kx^{k-1}, |x| < 1 = \frac{1}{(1-x)^2}$$

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = \ln(2)$$