Homework 8
Due November 21, 2013

CMSC 203

1. Section 7.1, Exercise 16 5 points
   What's the probability that a 5-card poker hand contains a flush, that is, five cards of the same suit?
   Ways to choose a suit: $C(4,1)$
   Ways to choose the 5 cards from that suit: $C(13,5)$
   Possible poker hands: $C(52,5)$
   Therefore, $p(\text{flush}) = \frac{C(4,1)C(13,5)}{C(52,5)} \approx 0.00198 \approx 0.2\%$

2. Section 7.1, Exercise 36 (*show the probability of both events*) 5 points
   Which is more likely: rolling a total of 8 when two dice are rolled or rolling a total of 8 when three dice are rolled?
   Probability of rolling 8 on two dice: 5 combinations with total 8, out of $36 = 5/36 = .13889$
   Probability of rolling 8 on three dice: 21 combinations with total 8, out of $216 = 21/216 = .09722$
   Therefore, rolling 8 on two dice is more likely

3. Section 7.2, Exercise 8a,c,e (*remember n is a variable and may appear in your answer*) 5 points each = 15 points
   What is the probability of these events when we randomly select a permutation of $\{1, 2, ..., n\}$ where $n \geq 4$

   a) 1 precedes 2
      1 either precedes 2, or 2 precedes 1. These are the only two possible case, and since both are equally likely, there is a 50% chance that 1 precedes 2.

   c) 1 immediately precedes 2
      There is an $\frac{n-1}{n}$ percent chance that 1 is not the last digit (since there are $n-1$ slots to choose from out of n total slots). For each of these cases, there is a $1/n-1$ percent chance that 2 is placed in the digit immediately after 2 (since there is 1 slot we need the 2, out of $n-1$ total slots). So the total chance of 1 immediately preceding 2 is the multiplication of these: $\frac{n-1}{n(n-1)} = \frac{1}{n}$

   e) n precedes 1 and n precedes 2 Similar to **a)**, except there are 3 possible suborderings of $1, 2,$ and n. Only one of has n as the first number, so the total probability is $\frac{1}{3}$

4. Section 7.2, Exercise 16 10 points
   Show that if E and F are independent events, then $\overline{E}$ and $\overline{F}$ are also independent events.
   If E and F are independent events, then $p(E \cap F) = p(E)P(F)$
   Complement: $p(\overline{E})p(\overline{F}) = (1 - p(E))(1 - p(F))$
   Distribute: $1 - p(E) - p(F) + p(E)p(F)$
   Substitute in $p(E)p(F) = p(E \cap F)$: $1 - p(E) - p(F) + p(E \cap F)$
   Factor out a negative: $-(p(E) + p(F) - p(E \cap F) - 1)$
   Use identity that $p(E) + p(F) - p(E \cap F) = p(E \cup F)$: $-(p(E \cup F) - 1) = 1 - p(E \cup F) = p(\overline{E} \cap \overline{F})$
   Therfore, $p(\overline{E})p(\overline{F}) = p(\overline{E} \cap \overline{F})$ so $\overline{E}$ and $\overline{F}$ are independent

5. Section 7.2, Exercise 24 *(modified)* 10 points
   What is the conditional probability that exactly four heads appear when a fair coin is flipped five times, given that the first flip came up tails? Show whether it the probability of heads appearing 4 times after 5 flips is dependent or independent of the first flip.
   E: Four heads appear when a coin is flipped five times
   F: The first flip is a tails
   $E = \{\text{HHHHT, HHHTH, HHTHH, HTHHH, THHHH}\}; |E| = 5$
   $|S| = 2^5 = 32$
   $p(E) = 5/32; p(F) = 1/2$ $p(E \cap F) = 1/32$ since $E \cap F = \{\text{THHHH}\}$
   $p(E)p(F) = 5/32 * 1/2 = 0.078125$
   $p(E)p(F) \neq p(E \cap F)$, so the events are dependent
   $p(E|F) = \frac{p(E \cap F)}{p(F)} = \frac{1/32}{1/2} = 0.0625$

6. Section 7.2, Exercise 32 5 points each = 15 points
   Find the probability that a randomly generated bit string of length 10 begins with a 1 or ends with a 00 for the following conditions:

   a) a 0 bit and a 1 bit are equally likely
      $p(E) = .5$
      $p(F) = .5 \cdot .5 = .25$
      $p(E \cap F) = .5 \cdot .5 \cdot .5 = .125$
      $p(E \cup F) = p(E) + p(F) - p(E \cap F) = .5 + .25 - .125 = .625$
   b) the probability that a bit is a 1 is 0.6
      $p(E) = .6$
      $p(F) = .4 \cdot .4 = .16$
      $p(E \cap F) = .6 \cdot .4 \cdot .4 = .096$
      $p(E \cup F) = p(E) + p(F) - p(E \cap F) = .6 + .16 - .096 = .664$
   c) the probability that the ith bit is a 1 is $1/2^i$ for $i = 1, 2, 3, \ldots, 10$
      $p(E) = .5$
      $p(F) = 1/2^9 \cdot 1/2^{10} = .0000019073$
      $p(E \cap F) = .5 \cdot 1/2^9 \cdot 1/2^{10} = .0000009537$
      $p(E \cup F) = p(E) + p(F) - p(E \cap F) = .5000009537$

7. Section 7.3, Exercise 10 5 points each = 20 points
   Suppose that 4% of the patients tested in a clinic are infected with avian influenza. Furthermore, suppose that when a blood test for avian influenza is given, 97% of the patients infected with avian influenza test positive and that 2% of the patients not infected with avian influenza test positive. What is the probability that...
   Let E be that the person has avian influenza and let F be that the test was positive.

   a) a patient testing positive for avian influenza with this test is infected with it?
      $p(E) = 4\%$, $p(F|E) = 97\%$, $p(F|\bar{E}) = 2\%$
      so $p(E|F) = \frac{p(F|E)p(E)}{p(F|E)p(E)+p(F|\bar{E})p(\bar{E})} = \frac{.97 \cdot .04}{.97 \cdot .04 + .02 \cdot .96}$
      $= .668$
   b) a patient testing positive for avian influenza with this test is not infected with it?
      $p(\bar{E}|F) = \frac{p(F|\bar{E})p(\bar{E})}{p(F|\bar{E})p(\bar{E})+p(F|E)p(E)} = \frac{.02 \cdot .96}{.02 \cdot .96 + .97 \cdot .04}$
      $= .331$
   c) a patient testing negative for avian influenza with this test is infected with it?
      $p(E|\bar{F}) = \frac{p(\bar{F}|E)p(E)}{p(\bar{F}|E)p(E)+p(\bar{F}|\bar{E})p(\bar{E})} = \frac{.03 \cdot .04}{.03 \cdot .04 + .98 \cdot .96}$
      $= .00127$
   d) a patient testing negative for avian influenza with this test is not infected with it?
      $1 - p(E|\bar{F}) = .9987$

8. Section 7.3, Exercise 18 10 points
   Suppose that a Bayesian spam filter is trained on a set of 500 spam messages and 200 messages that are not spam. The word "exciting" appears in 40 spam messages and in 25 message that are not spam. Would an incoming message be rejected as spam if it contains the word "exciting" and the threshold for rejecting spam is 0.9?
   E: Spam message; F: the word "exciting" appears
   $p(E) = 500/700 = 5/7$
   $p(F|E) = \frac{p(F \cap E)}{p(E)} = 40/500 = 2/25$
   $p(\bar{E}) = 1 - p(e) = 2/7$
   $p(F|\bar{E}) = \frac{p(F \cap \bar{E})}{p(\bar{E})} = \frac{25}{200} = 1/8$
   $p(E|F) = \frac{p(F|E)p(E)}{p(F|E)p(E)+p(F|\bar{E})p(\bar{E})} = \frac{(2/25)*(5/7)}{((2/25)*(5/7))+((1/8)*(5/7))} = 16/41 \approx 0.39$
   This is **not** over the threshold of 0.9, so a message containing the word "exciting" is not rejected as spam.

9. Section 7.4, Exercise 6 5 points
   What is the expected value when a \$1 lottery ticket is bought in which the purchaser wins exactly \$10 million if the ticket contains the six winning numbers chosen from the set $\{1, 2, 3, \ldots, 50\}$ and the purchaser wins nothing otherwise?
   $p(\text{win}) = 1/C(50, 6) = 1/15890700$

$V(\text{win}) = p(\text{win}) \cdot 10000000 = .629$
$V(\text{lose}) = (1 - p(\text{win}) \cdot 0 = 0$
$V(\text{play}) = V(\text{win}) + V(\text{lose}) - 1 = -.371$

10. Chapter 7.4, Exercise 10 5 points
Suppose that we flip a fair coin until either it comes up tails twice or we have flipped it six times. What is the expected number of times we flip the coin?
$p(1) = 0$
$p(2) = .25$
$p(3) = .25$
$p(4) = .1875$
$p(5) = .125$
$p(6) = .1875$
$V = 2p(2) + 3p(3) + 4p(4) + 5p(5) + 6p(6) = 3.75$