1 Complexity (15 points)

0. You are going to a very large party, and are provided a list of outfits everyone will be wearing. Assuming an outfit consist of a shirt and a pair of pants, write an algorithm that will search your drawer of shirts and your drawer of pants for a outfit combination that nobody else will be wearing. Write your answer in **pseudocode**. You may assume outfits can somehow be ordered. (10)

   Any valid procedure will work here, as long as – at minimum – you compare every outfit in your closet versus every outfit at the party. It should be possible that your shirt and pants combo are worn by two distinct people, but the outfit itself is unique. A simple procedure may be:

   function CHECKOUTFIT($shirts$:my shirts, $pants$:my pants, $partyfits$:outfits at party)
       for $s \in shirts$ **do**
           for $p \in pants$ **do**
               for $o \in partyfits$ **do**
                   if $s + p \neq o$ **then**
                       return $s + p$
       return False
   end function

   Alternatively, you may use a more complex solution for better runtime, which are explained below.

1. What is the asymptotic analysis (big-O) of the procedure you gave above? (5)

   In the pseudocode above, the runtime is $O(n^3)$. There are faster ways to perform this procedure, however. If you generate a list of outfits in your closet, you can perform a binary search on the party outfits for a total of $O(n^2 \log n)$. Alternatively, you can store outfits into a hash table, and the lookup can be performed in constant time for a total run-time of $O(n^2)$.

2. If you wanted to sort your clothes, what sort would you use and why? (5)

   Any valid sort and reasoning works, as long as it isn't bubble sort, and you explained your answer a little. For example, insertion or merge sort.

2 Induction (20 points)

3. Which amounts of money can be formed using just two-dollar bills and five-dollar bills? (5)

   All denominations except \$1 and \$3

4. Prove your answer above using strong induction. (15)

   $P(n)$ is the statement that we can form n dollars just using 2-dollar and 5-dollar bills.

   Basis Step

   $P(2) = \$2$

   $P(4) = 2 * \$2$

   $P(5) = \$5$

   $P(6) = 3 * \$2$

   Inductive Step

   $P(j)$ is true for all j with $5 \leq j \leq k$, where $k \geq 6$

   Show $P(k+1)$ is true

   Since $k - 1 \geq 5$, we know $P(k-1)$ is true. If we add a \$2 to $P(k-1)$ we can achieve $P(k+1)$

   This concludes the inductive step.

   This proves $P(n)$ using strong induction

3 Recursion (15 points)

5. Develop a recursive algorithm for finding the sum of the first n odd positive integers. (5)

Any valid recursive function will work here, as long as it gives an appropriate answer. It is helpful to design an algorithm that mathematically computes each square number, rather than attempting to pass a counter and the current sum. To test, the output should be equal to the input squared. An optimal solution:

```
function ODDSUM(n: the number of odd positive integers to add)
    if n = 1 then
        return 1
    else
        return OddSum(n − 1) +2n − 1
end function
```

6. Prove the recursive algorithm you found above is correct. (10)

This is dependent on your previous answer in many cases, but in essentially all cases, an inductive proof should be given. If a basis step is given, along with proof that some $P(k+1)$ value is correct if $P(k)$ is correct, then full points are given.

Given $P(k)$ is the sum of the first k odd positive integers, prove $P(k+1)$ is the sum of the first $k+1$ odd positive integers.

Basis Step:

$\text{OddSum}(1) = 1$

Inductive Step:

When calculating $P(k+1)$, since $k > 1$, we return $\text{OddSum}(k) + 2(k+1) - 1$

We know $\text{OddSum}(k)$ is correct sum of the first k odd integers, and $2(k+1) - 1$ is the $(k+1)^{\text{th}}$ odd integer.

Thus, our function is calculating $(1 + 3 + \ldots + (2k-1)) + (2(k+1) - 1)$, which is the sum of the first $k+1$ odd integers.

This ends our inductive hypothesis, and proves our algorithm is correct using inductive hypothesis.

4 Counting (25 points)

7. Show that among any group of five (not necessarily consecutive) integers, there are two with the same remainder when divided by 4. (10)

First, we know there are 4 distinct categories for some integers based on its remainder when divided by 4: $\{0, 1, 2, 3\}$. Therefore, $k = 4$.

We are selecting 5 integers, so $N = 5$.

Pigeonhole principle tells us that $\lceil \frac{N}{k} \rceil \geq r \rightarrow \lceil \frac{5}{4} \rceil = 2$

Therefore, there must be at least 2 integers that have the same remainder.

8. A club has 10 members. How many ways are there to choose a president, vice president, secretary, and treasurer of the club, where no person can hold more than one office? (5)

We will use a permutation with $n = 10$ and $r = 4$ (since we are selecting 4 positions from 10 slots). A permutation is used since the order we select the members matters, since there are ranks to assign.

$\frac{10!}{(10-4)!} = \frac{10!}{6!} = 5040$

9. How many students are in a class if there are 38 computer science majors, 23 mathematics majors, and 7 joint majors (assuming the class consists of no other majors, and a student is still considered a computer science or mathematic major if they are joint majoring)? (5)
We will use the difference rule: $38CS + 23MATH - 7BOTH = 54students$

10. How many ways are there to select 3 candies from a bag of Snickers, Butterfingers, Twix bars, Tootise Rolls, and Almond Joys (assuming you like all these candies)? (5)
We will use combination with repetition, since we may select multiples of the same candy (and order does not matter).
There are 5 types of candies, so $n = 5$. We may select 3 candies, so $r = 3$.
$\frac{(n+r-1)!}{r!(n-1!)} = \frac{(5+3-1)!}{3!(5-1)!} = \frac{7!}{3!4!} = 35$

5 Binomial Expansion (15 points)

11. Using **Pascal's Triangle**, and find the complete binomial expansion of $(x+y)^8$. Include your diagram of Pascal's Triangle and the full expansion. (10)
Valid triangle should go here.
The result should be: $x^8+8x^7y+28x^6y^2+56x^5y^3+70x^4y^4+56x^3y^5+28x^2y^6+8xy^7+y^8$

12. Find the coefficient of x^8y^2 for $(x+y)^{10}$. (5)
$\binom{10}{8}=\binom{10}{2}=\frac{10!}{2!8!}=45$
You may also expand the triangle above to reach the same conclusion.