CMSC 203
Exam 1
October 3, 2013

If you find yourself spending too long on a problem, skip it and move on to the next one. For all problems, partial credit will be given for incorrect answers only if an explanation is included.

Page	Points	Score
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	15	
Total:	90	

1 Propositional Logic (10 points)

0. Produce a truth table for the following propositional statement: (8)
$((p \land q) \to r) \iff (p \to (q \to r))$

p	q	r	$p \land q$	$(p \land q) \to r$	$q \to r$	$p \to (q \to r)$	$((p \land q) \to r) \iff (p \to (q \to r))$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	F	T	T	T	T
F	T	F	F	T	F	T	T
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T

1. Is this either a tautology or a contradiction? (2)
Yes, it is a tautology since the statement resolves to "True" regardless of the inputs.

2 Predicate Logic (20 points)

Let P, Q, R, and S be the following predicates:
$P(x)$: x is a rock
$Q(x)$: x sinks in water
$R(x)$: x is very small
$S(x)$: x can be used to build a bridge

2. Convert the following english sentences to logical sentences in terms of P, Q, R, S, logical connectives, and quantifiers: (10)

- All rocks can be used to build a bridge.
- There is nothing very small which sinks in water
- Anything which is **not** very small cannot be used to build a bridge.
- The Blarney Stone is a rock

1. $\forall x(P(x) \rightarrow S(x))$
2. $\neg\exists x(R(x) \wedge Q(x)) \equiv \forall x \neg(R(x) \wedge Q(x))$
3. $\forall x(\neg R(x) \rightarrow \neg S(x))$
4. $P(BlarneyStone)$ or $BlarneyStone(x) \rightarrow P(x)$

3. Given the sentences above, show that the Blarney Stone floats in water (doesn't sink) using rules of inference and logical equivalences. Show each new sentence in each step of your proof, and apply exactly one rule of inference or equivalence per step. You are welcome to label each rule for each step. (10)

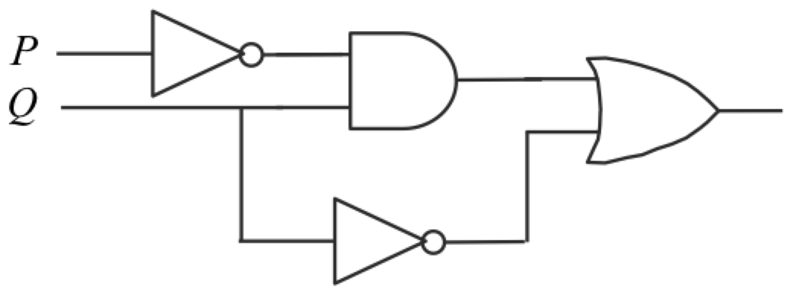
There are two ways to prove, given assumptions on previous page.
If BlarneyStone is a constant, we can show that $\neg Q(BlarneyStone)$

1. $P(BlarneyStone)$ (Given assumption 4)
2. $P(x) \rightarrow S(x)$ (Given assumption 1)
3. $S(BlarneyStone)$ (Modus ponens of 1 and 2)
4. $\neg R(x) \rightarrow \neg S(x)$ (Given assumption 3)
5. $S(x) \rightarrow R(x)$ (Logical equivalence [contrapositive] of 4)
6. $R(BlarneyStone)$ (Modus ponens of 3 and 5)
7. $\neg(R(x) \wedge Q(x))$ (Given assumption 2)
8. $\neg R(x) \vee \neg Q(x)$ (De Morgan's of 7)
9. $\neg R(BlarneyStone) \vee \neg Q(BlarneyStone)$
10. $\neg Q(BlarneyStone)$ (Identity Law and 6)

Alternatively, we can show $BlarneyStone(x) \rightarrow \neg Q(x)$

1. $BlarneyStone(x) \rightarrow P(x)$ (Given assumption 4)
2. $P(x) \rightarrow S(x)$ (Given assumption 1)
3. $BlarneyStone(x) \rightarrow S(x)$ (Hypothetical syllogism of 1 and 2)
4. $\neg R(x) \rightarrow \neg S(x)$ (Given assumption 3)
5. $S(x) \rightarrow R(x)$ (Logical equivalence [contrapositive] of 4)
6. $BlarneyStone(x) \rightarrow R(x)$ (Hypothetical syllogism of 3 and 5)
7. $\neg(R(x) \wedge Q(x))$ (Given assumption 2)
8. $\neg R(x) \vee \neg Q(x)$ (De Morgan's of 7)
9. $R(x) \rightarrow \neg Q(x)$ (Logical equivalence of 8)
10. $BlarneyStone(x) \rightarrow \neg Q(x)$ (Hypothetical syllogism of 6 and 9)

3 Boolean Algebra (10 points)



4. Write the equivalent logic statement represented by the digital logic circuit above. (5)
   $(\neg P \land Q) \lor \neg Q$

5. Simplify the logic statement by minimizing the number of connectives used. (5)
   $(\neg P \land Q) \lor \neg Q$ (Given)
   $(\neg Q \lor \neg P) \land (\neg Q \lor Q)$ (Distributive)
   $(\neg Q \lor \neg P) \land T$ (Negation)
   $(\neg Q \lor \neg P)$ (Identity)
   $\neg(Q \land P)$ (De Morgan's)

4 Proofs (20 points)

6. Prove or disprove that x^3 is irrational, then x is irrational. What kind of proof did you use? (10)

p: x^3 is irrational
q: x is irrational

Proof by contradiction: $\neg q \rightarrow \neg p$
Assume $\neg q$: x is rational
$x = a/b$, where $b \neq 0$ and a and b share no common factors
$x^2 = a^2/b^2$ is rational, since a^2 and b^2 will share the same factors as a and b respectively
Therefore, $x^3 = a^3/b^3$ is still rational
Since, x^3 is rational, $\neg q$
This shows $\neg q \rightarrow \neg p$ which is equivalent to $p \rightarrow q$

7. Prove that if $n \in \mathbb{Z}$, then $n^2 + 3n + 5$ is an odd integer. *(Hint: n may be even or odd)* (10)

p: $n \in \mathbb{Z}$
q: $n^2 + 3n + 5$ is an odd integer

Proof by cases: $p \to q \equiv (p_1 \to q) \wedge (p_2 \to q) \wedge (p_3 \to q)$
p_1: n is an even number, p_2: n is an odd number, p_3: n is 0

Assume n is an even number

$$
\begin{aligned}
n &= 2k \\
n^2 + 3n + 5 &= (2k)^2 + 3(2k) + 5 \\
&= 4k^2 + 6k + 4 + 1 \\
&= 2(2k^2 + 3k + 2) + 1 \\
&= 2k\prime + 1 \\
&= odd
\end{aligned}
$$

Therefore, if n is an even number, $n^2 + 3n + 5$ is odd

Assume n is an odd number

$$
\begin{aligned}
n &= 2k + 1 \\
n^2 + 3n + 5 &= (2k + 1)^2 + 3(2k + 1) + 5 \\
&= (4k^2 + 4k + 1) + (6k + 3) + 5 \\
&= 4k^2 + 10k + 8 + 1 \\
&= 2(2k^2 + 5k + 4) + 1 \\
&= 2k\prime + 1 \\
&= odd
\end{aligned}
$$

Therefore, if n is an odd number, $n^2 + 3n + 5$ is odd

Assume n is 0

$$
\begin{aligned}
n &= 0 \\
0^2 + 3 * 0 + 5 &= 0 + 0 + 5 \\
&= 5 \\
&= odd
\end{aligned}
$$

Therefore, if n is 0, $n^2 + 3n + 5$ is odd

Since $(p_1 \to q) \wedge (p_2 \to q) \wedge (p_3 \to q)$, $p \to q$

5 Sets (10 points)

8. Prove $\overline{A \cup B \cup C} = \overline{A} \cap \overline{B} \cap \overline{C}$ without using Venn Diagrams. (10)

   1. $\{x | x \in \overline{(A \cup B \cup C)}\}$
   2. $\{x | x \notin (A \cup B \cup C)\}$
   3. $\{x | x \notin A \vee x \notin B \vee x \notin C\}$
   4. $\{x | x \in \overline{A} \vee x \in \overline{B} \vee x \in \overline{C}\}$
   5. $\{x | x \in \overline{A} \cap \overline{B} \cap \overline{C}\}$

6 Functions (15 points)

For the following questions, list whether it is **one-to-one**, **onto**, or a **one-to-one correspondence** (bijection).

9. $f(x) = 9x/12$ $f(x) : \mathbb{R} \to \mathbb{R}$ (3)
   One-to-One? Yes Onto? Yes Bijection? Yes

10. $f(x) = \lceil x \rceil * 2$ $f(x) : \mathbb{R} \to \mathbb{Z}$ (3)
    One-to-One? No Onto? No Bijection? No

11. $f(x) = (x^2 + 1)/(x^2 + 2)$ $f(x) : \mathbb{R} \to \mathbb{R}$ (3)
    One-to-One? No Onto? No Bijection? No

12. $f(x) = x^7 + 13$ $f(x) : \mathbb{R} \to \mathbb{R}$ (3)
    One-to-One? Yes Onto? Yes Bijection? Yes

13. $f(x, y) = x^2 - y$ $f(x, y) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ (3)
    One-to-One? No Onto? Yes Bijection? No