CMSC 203 – Sample Proofs

1 Proof by Mathematical Induction

Prove by mathematical induction that

$$2^n \leq n!$$

for $n > 3$.

PROOF.

Basis step. When $n = 4$, $2^n = 2^4 = 16$, and $n! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$. Therefore, $2^n \leq n!$ in the base case.

Inductive step. Suppose that $2^n \leq n!$, for some $n > 4$. Then $2^{n+1} \leq (n+1)!$.

Proof of inductive step.

$$\begin{array}{rcll} 2^{n+1} & = & 2 \cdot 2^n & \\ & \leq & 2 \cdot n! & \text{By the inductive hypothesis.} \\ & \leq & (n+1) \cdot n! & \text{Since } n > 4,\ 2 < n+1. \\ & = & (n+1)! & \end{array}$$

Therefore, $2^n \leq n!$ for all $n > 3$.

Q.E.D.

2 Proof by Logical Equivalences

Prove by logical equivalences that

$$\neg(p \wedge q) \vee \neg(q \wedge r) \Leftrightarrow p \rightarrow \neg(q \wedge r)$$

PROOF.

$$\begin{array}{rcll} \neg(p \wedge q) \vee \neg(q \wedge r) & \Leftrightarrow & (\neg p \vee \neg q) \vee (\neg q \vee \neg r) & \text{By DeMorgan's Law} \\ & \Leftrightarrow & \neg p \vee \neg q \vee \neg r & \text{By associativity of disjunction} \\ & \Leftrightarrow & \neg p \vee \neg(q \wedge r) & \text{By DeMorgan's Law} \\ & \Leftrightarrow & p \rightarrow \neg(q \wedge r) & \text{By definition of implication} \end{array}$$

Q.E.D.

3 Indirect Proof

Prove using an indirect proof that all primes greater than 2 are odd.

PROOF. Suppose that p is an even prime greater than 2. By definition of evenness, $2|p$, i.e., $p = 2n$ for some n, and $n|p$. Since $p > 2$, n must be greater than 1. But a prime number is, by definition, only divisible by 1 and itself. Therefore, no even prime greater than 2 can exist.

Q.E.D.

4 Proof by Cases

Prove using an argument by cases that

$$n \ mod\ 2 \leq \ mod\ 4$$

PROOF. Let $P(n)$ represent for the proposition being proved. Consider four cases: n mod $4 = 0, 1, 2, 3$.

When n mod $4 = 0$, n mod $2 = 0$. $0 \leq 0$, so $P(n)$ holds.
When n mod $4 = 1$, n mod $2 = 1$. $1 \leq 1$, so $P(n)$ holds.
When n mod $4 = 2$, n mod $2 = 0$. $0 \leq 2$, so $P(n)$ holds.
When n mod $4 = 3$, n mod $2 = 1$. $1 \leq 3$, so $P(n)$ holds.

Q.E.D.