Solving Recurrences with Generating Functions: A Tutorial

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A paper submitted to the
Computer Science Department
in partial fulfillment of the requirements for
CMSC-693 Research and Writing Skills
at the University of Maryland Baltimore County

May 19, 1993

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Abstract

Generating functions are a powerful technique for solving recurrences. Recurrences are important in the analysis of algorithms because the running time of an algorithm can be estimated by solving the corresponding recurrence. As a tutorial, we present a method for solving recurrences with generating functions. In addition, as an application to cryptography, we explain how generating functions can be used to analyze linear shift registers. We introduce probability generating functions through an application involving a queuing network model.

Generating functions provide a method for determining the terms of many sequences. The method for solving a recurrence with generating functions has three steps: 1) Choose the general form of the generating function $G(x)$, 2) Derive a formula for $G(x)$ using the recurrence, 3) Derive a closed form expression for the coefficients of $G(x)$. This closed form expression allows us to compute the nth term in the recurrence. Generating functions apply wherever sequences apply. In cryptography, a linear shift register (LSR) computes a sequence S of bits as it encodes a message. We explain how the period of S is related to the generating function of S. Generating functions can also be used to compute moments of probability distributions. Such generating functions are called probability generating functions. Probability generating functions are discussed and applied to a queuing network model. In addition, we include an appendix of generating function exercises and solutions.

Keywords. Generating functions, recurrences, sequences, power series, Fibonacci numbers, linear shift registers, queuing network model.

Contents

List of Tables

1 Introduction

How can the solution to a recurrence for a given algorithm be calculated? For example, how can we solve the recurrence for mergesort: $t(n) = 2t(n/2) + n$, with $t(1) = 1$? Many techniques exist for solving recurrences: iteration, characteristic equations, substitution, change of variable, and the master method. Generating functions are a more powerful technique for solving recurrences which allow us to use the power of calculus. Recurrences have the property that the nth term in the sequence is a function of previous terms. With generating functions, the nth term in the sequence is the coefficient of the nth term in the generating function.

According to Knuth [6], the method of generating functions was introduced in the work of DeMoivre around 1720. In 1748 Euler developed the method in connection with partition problems. In the late eighteenth and early nineteenth century Laplace developed the techniques further in connection with probability theory.

I present a tutorial on generating functions, with the emphasis on solving recurrences. Section 2 explains what generating functions are, Section 3 discusses how we can use generating functions to solve recurrences, and Section 4 presents operations for manipulating them. Because generating functions are closely related to sequences, they can apply wherever sequences are used. Section 5 shows how generating functions apply to linear shift registers in cryptography. Probability generating functions are another type of generating function. Section 6 introduces probability generating functions and presents the queuing network model as an application. This paper is intended for graduate students as an introduction to generating functions and their application to recurrences. The tutorial's novel contribution is its ability to serve as a good reference on generating functions for students in the computer science department.

2 What is a Generating Function?

We can represent any infinite sequence of numbers $(g_n) = g_0, g_1, g_2, \ldots$ as an infinite power series with an auxiliary variable x as follows

$$G(x) = g_0 + g_1 x + g_2 x^2 + \cdots = \sum_{n=0}^{\infty} g_n x^n. \tag{1}$$

We now try to obtain information about the function G, which is a single quantity representing the entire infinite sequence (g_n). For each $n = 0, 1, 2, \ldots$, g_n is the coefficient of the term x^n in $G(x)$.

$G(x)$ is called the *generating function* for the sequence (g_n). Generating functions provide a simple way to "encode" a sequence, which can be "decoded" to find the terms of the sequence. Once we have the generating function, we can determine any term in the sequence.

Many people who use generating functions do not concern themselves with the question of convergence of generating functions since they can check the solution. We are only looking at possible approaches to the solution to a problem. The original problem domain is transformed to that of generating functions in the hope that the problem is easier to solve in the new domain. The problem is eventually transformed to the original problem domain in order to obtain the solution. Once we have a solution it is possible to justify the solution independently with the principle of induction. Here, the original problem is to compute a closed form expression for a given recurrence relation.

3 A Method for Solving Recurrences

One of the important uses of generating functions is in the solution of recurrence relations. Given a sequence (g_n) that satisfies a given recurrence, we want a closed form expression for g_n in terms of n. We want to be able to calculate the nth term in the recurrence by using the closed form expression g_n. Because this problem is nontrivial, we transform the problem into a new one in which we solve for the generating function $G(x)$. Using the coefficients of $G(x)$ we then return to the problem of solving for the closed form of g_n in terms of n. A three-step procedure solves recurrences using generating functions:

1) Choose the general form of the generating function $G(x)$.

2) Using the recurrence, derive a formula for $G(x)$ in terms of x.

3) Derive a closed form expression, g_n, for the coefficients of $G(x)$.

Example: Fibonacci Numbers
The *Fibonacci numbers* are defined by the recurrence

$$f_n = \begin{cases} f_{n-1} + f_{n-2} & n \geq 2 \\ 1 & n = 1 \\ 0 & n = 0 \end{cases} \tag{2}$$

Given the recurrence for the Fibonacci numbers, we want to apply the three steps in order to get a closed form expression for f_n.

Step 1:
Choose the general form of $G(x) = \sum_{n\geq 0} f_n x^n$. We choose this form for $G(x)$ because we need the generating function for Eq. 2 to have the form of the generating function in Eq. 1. This form allows us to calculate the generating function $G(x)$ of the recurrence shown in Step 2.

Step 2:
Using Eq. 2, we derive a formula for $G(x)$ in terms of x. To do this we multiply both sides of the equation Eq. 2 by x^n and sum for all $n \geq 2$ to formulate

$$\sum_{n=2}^{\infty} f_n x^n = \sum_{n=2}^{\infty} f_{n-1} x^n + \sum_{n=2}^{\infty} f_{n-2} x^n. \tag{3}$$

Now we make each sum resemble the generating function $G(x)$. Replace the n in the second sum with $n+1$ and the n in the third sum by $n+2$ to obtain

$$\sum_{n=2}^{\infty} f_n x^n \quad = \quad \sum_{n+1=2}^{\infty} f_n x^{n+1} + \sum_{n+2=2}^{\infty} f_n x^{n+2} \tag{4}$$

$$= \quad x\sum_{n=1}^{\infty} f_n x^n + x^2 \sum_{n=0}^{\infty} f_n x^n. \tag{5}$$

We need each sum to have a lower limit of zero, so we modify Eq. 5 by adding and subtracting the necessary cancelling terms. Because the second summation is multiplied by x, each cancelling term must be multiplied by x:

$$f_0 + f_1 x + \left(\sum_{n=2}^{\infty} f_n x^n\right) - f_0 - f_1 x = f_0 x + x\left(\sum_{n=1}^{\infty} f_n x^n\right) - f_0 x + x^2 \sum_{n=0}^{\infty} f_n x^n. \tag{6}$$

By adding the positive terms to their corresponding summations, we can rewrite Eq. 6 as

$$G(x) - f_0 - f_1 x = xG(x) - f_0 x + x^2 G(x). \tag{7}$$

Using the initial conditions, the formula for $G(x)$ becomes

$$G(x) = \frac{x}{1 - x - x^2}. \tag{8}$$

Step 3:
To derive a closed form expression f_n for the coefficients of $G(x)$ we must first determine what those coefficients are. To do this, we expand $G(x)$ into a power series. Start by factoring the denominator and finding the roots of the equation. The roots are $a = (-1+\sqrt{5})/2$ and $b = (-1-\sqrt{5})/2$. We require that $\alpha = 1/a \approx 1.6180$ and $\beta = 1/b \approx -0.6180$. Because the present form of $G(x)$ is not a common generating function, we simplify it to a combination of common generating functions. We expand $G(x)$ by using partial fraction decomposition:

$$G(x) = \frac{x}{(1-\frac{x}{a})(1-\frac{x}{b})} = \frac{A}{1-\frac{x}{a}} + \frac{B}{1-\frac{x}{b}}. \tag{9}$$

We calculate $A = 1/\sqrt{5}$ and $B = -1/\sqrt{5}$, therefore

$$G(x) = \frac{1}{\sqrt{5}}\left(\frac{1}{1-\alpha x} - \frac{1}{1-\beta x}\right). \tag{10}$$

To get the power series for $G(x)$, we use the fact that $1/(1-\alpha x)$ equals the infinite sum $1 + \alpha x + \alpha^2 x^2 + \cdots$. We substitute the infinite sum into Eq. 10 such that

$$G(x) = \frac{1}{\sqrt{5}}(1 + \alpha x + \alpha^2 x^2 + \cdots - 1 - \beta x - \beta^2 x^2 - \cdots). \tag{11}$$

By definition of $G(x)$, the coefficient of x^n is

$$f_n = \frac{(\alpha^n - \beta^n)}{\sqrt{5}}. \tag{12}$$

Now we have the closed form expression f_n for the coefficients of $G(x)$. Since the absolute value of β is less than 1, the β^n has very little effect for large n. For large n, $f_n \approx \alpha^n/\sqrt{5}$. For more details on the above calculations see Knuth [6].

4 Manipulation Operations

We can perform operations with generating functions. By manipulating generating functions we compute new generating functions. Most operations, if not all, can be justified without regard to convergence of the infinite power series. Below are some useful Maclaurin expansions that will be used in examples taken from Roberts [10]:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots, \text{for } |x| < 1, \tag{13}$$

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, \text{for } |x| < \infty, \tag{14}$$

Let us consider the generating functions $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0}^{\infty} b_n x^n$. Typical operations using $A(x)$ and $B(x)$ are shown in Table 1. In solving recurrences, we need the generating functions in the equations in the form of Eq. 1 in order to perform the necessary calculations to compute the formula for $G(x)$ and derive the closed form expression of the recurrence. When we used the cancelling terms in the second step for the Fibonacci recurrence, we were shifting those summations whose lower limit was not zero in order to make their lower limit zero and therefore in the form of Eq. 1. Table 2 lists some common generating functions along with their corresponding closed forms and sequences.[1] Now we present examples using some of the operations shown in Table 1. The first four examples describe techniques that are presented by Knuth [6].

Example 1: Addition
The sum of two generating functions is the addition of their coefficients corresponding to each power of x. Each coefficient of x is 1 in Eq. 13, and is $1/n!$ in Eq. 14. Using Eqs. 13 and 14, we see that their sum is:

$$\begin{aligned}
\frac{1}{1-x} + e^x &= (1 + x + x^2 + x^3 + \cdots) + \left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots\right) \\
&= (1+1) + (1+1)x + \left(1 + \frac{1}{2!}\right)x^2 + \left(1 + \frac{1}{3!}\right)x^3 + \cdots \\
&= \sum_{n=0}^{\infty} \left(1 + \frac{1}{n!}\right) x^n.
\end{aligned} \tag{15}$$

Example 2: Multiplication

The product of two generating functions is found by multiplying the second generating function by each term of the first generating function. The multiplication of two generating functions is also referred to as a *convolution* of the two generating functions. Using Eqs. 13 and 14, we see that their product is:

$$\frac{1}{1-x} e^x = (1 + x + x^2 + x^3 + \cdots) \left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots\right)$$

[1]The notation $[m \backslash n]$ equals 1 when m divides n, and 0 otherwise.

$$\begin{aligned} &= 1\left(1+x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\cdots\right)+x\left(1+x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\cdots\right) \\ &\quad +x^2\left(1+x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\cdots\right)+\cdots \\ &= 1+2x+\frac{5}{2}x^2+\frac{5}{3}x^3+\cdots. \end{aligned} \tag{16}$$

Example 3: Differentiation
If a power series $A(x) = \sum_{n\geq 0} a_n x^n$ is any power series that converges for at least one value of x with $|x| > 1$, then the derivative of $A(x)$, namely $A'(x)$, will also have this property. We can differentiate and integrate a power series $A(x)$. Therefore, we can do the same with generating functions. The derivative and antiderivative are calculated term by term. The derivative of Eq. 13 computes another generating function. Since

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left[\frac{1}{1-x}\right], \tag{17}$$

we see that

$$\begin{aligned} \frac{1}{(1-x)^2} &= \sum_{n=0}^{\infty} n x^{n-1} \\ &= \sum_{n=0}^{\infty}(n+1)x^n \\ &= 1+2x+3x^2+4x^3+\cdots. \end{aligned} \tag{18}$$

Example 4: Shifting
If $A(x)$ is the generating function for the sequence $(a_0, a_1, \ldots)$, then $x^k A(x)$ is the generating function for $(0, \ldots, 0, a_0, a_1, \ldots)$. We have essentially shifted the sequence k positions to the right. Take the generating function for e^x in Eq. 14, whose sequence is shown in Table 2. If we multiply e^x by x, then xe^x is the generating function for the sequence $(0, 1, 1, 1/2!, 1/3!, \ldots)$.

Likewise, we can subtract the first k terms from $A(x)$ and then proceed to divide by x^k to obtain $(A(x) - a_0 - a_1 x - \cdots - a_{k-1}x^{k-1})/x^k$ as the generating function for the sequence $(a_k, a_{k+1}, \ldots)$. We have shifted the sequence k positions to the left. Again we take the generating function for e^x. We use the generating function $(e^x - 1)/x$ to get the sequence $(1, 1/2!, 1/3!, \ldots)$.

Example 5: Short Cuts
Sometimes there are short-cut techniques that can be performed in order to get the generating function of a sequence. One technique is to use differencing. Another is to relate the generating function to geometric progressions. These techniques are described by Hillman [5].

Let $G(x) = g_0 + g_1 x + g_2 x^2 + \cdots$. Then $(1-x)G(x)$ is the power series $H(x) = h_0 + h_1 x + h_2 x^2 + \cdots$ where $h_0 = g_0$, $h_n = g_n - g_{n-1}$ for all $n \in \mathbb{N}$. This technique is known as *differencing*. As an example, let $G(x) = 1 + x + x^2 + \cdots$. We calculate $G(x)$ multiplied by $-x$ and then add

Operation	Notation	Generating Function
Addition	$A(x) + B(x)$	$\sum_{n\geq 0}(a_n + b_n)x^n$
Multiplication	$A(x)B(x)$	$\sum_{n\geq 0}(a_n x^n \sum_{n\geq 0} b_n x^n)$
Division	$A(x)/B(x)$	$\sum_{n\geq 0}(a_n x^n / \sum_{n\geq 0} b_n x^n)$
Differentiation	$\frac{d}{dx}A(x)$	$\sum_{n\geq 0} n a_n x^{n-1}$
Integration	$\int_0^x A(t)dt$	$\sum_{n\geq 0} \frac{1}{n+1} a_n x^{n+1}$
Shifting	$x^k A(x)$	$\sum_{n\geq k} a_{n-k} x^n$
	$x^{-k} A(x)$	$\sum_{n\geq 0} a_{n+k} x^n$
Differencing	$(1-x)A(x)$	$a_0 + \sum_{n\geq 1}(a_n - a_{n-1})x^n$

Table 1: Operations for Manipulating Generating Functions

the resulting equation to $G(x)$ in order to calculate the closed form of $G(x)$:

$$\begin{aligned} G(x) &= 1 + x + x^2 + \cdots \\ -xG(x) &= \quad - x - x^2 - \cdots \\ (1-x)G(x) &= 1 + (1-1)x + (1-1)x^2 + \cdots = 1 \\ G(x) &= 1/(1-x). \end{aligned} \tag{19}$$

From the Eq. 19, we see that the closed form for $1 + x + x^2 + \cdots$ is $(1-x)^{-1}$. From the coefficients of the power series, the sequence becomes $(1, 1, 1, \ldots)$.

The second and fifth generating functions in Table 2 are related to *geometric progressions*. As an example, $3/1 - 2x$ is the generating function for the geometric progression $3, 6, 12, 24, \ldots, 3 \cdot 2^n, \ldots$. The generating function for the sequence $g_n = 5(n+1)3^n$ is $5/(1-3x)^2$.

Generating Function	Closed Form	Sequence
$\sum_{n\geq 0} x^n$	$\frac{1}{1-x}$	$(1, 1, 1, 1, \ldots)$
$\sum_{n\geq 0} c^n x^n$	$\frac{1}{1-cx}$	$(1, c, c^2, c^3, \ldots)$
$\sum_{n\geq 0} (-1)^n x^n$	$\frac{1}{1+x}$	$(1, -1, 1, -1, \ldots)$
$\sum_{n\geq 0} (n+1) x^n$	$\frac{1}{(1-x)^2}$	$(1, 2, 3, 4, \ldots)$
$\sum_{n\geq 0} (n+1) c^n x^n$	$\frac{1}{(1-cx)^2}$	$(1, 2c, 3c^2, 4c^3, \ldots)$
$\sum_{n\geq 0} [2\backslash n] x^n$	$\frac{1}{1-x^2}$	$(1, 0, 1, 0, \ldots)$
$\sum_{n\geq 0} [k\backslash n] x^n$	$\frac{1}{1-x^k}$	$(1, 0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots)$
$\sum_{n\geq 1} \frac{1}{n} x^n$	$\ln \frac{1}{1-x}$	$\left(0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right)$
$\sum_{n\geq 1} \frac{(-1)^{n+1}}{n} x^n$	$\ln(1+x)$	$\left(0, 1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots\right)$
$\sum_{n\geq 0} \frac{1}{n!} x^n$	e^x	$\left(1, 1, \frac{1}{2!}, \frac{1}{3!}, \frac{1}{4!}, \ldots\right)$
$\sum_{n\geq 0} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$	$\sin x$	$\left(1, -\frac{1}{3!}, \frac{1}{5!}, -\frac{1}{7!}, \ldots\right)$
$\sum_{n\geq 0} \binom{k}{n} x^n$	$(1+x)^k$	$x_n = \binom{k}{n}$
$\sum_{n\geq 0} \binom{k+n-1}{n} x^n$	$\frac{1}{(1-x)^k}$	$x_n = \binom{k+n-1}{n}$

Table 2: Common Generating Functions, Closed Forms and Sequences.

5 Generating Functions and Feedback Shift Registers

Generating functions are useful in cryptology for describing sequences of states produced by feedback shift registers. Cryptography uses different types of ciphers for encoding messages. The *linear shift register* is one such cipher. The linear shift register (LSR) R has n stages and an initial fill $x_0 \neq 0$. R computes a sequence of bits $S = (s_0, s_1, \ldots)$.

Before explaining how linear shift registers relate to generating functions, we must state some definitions and notations:[2] $\mathbb{Z}_2$ is the finite field of two elements, namely 0 and 1. $\mathbb{Z}_2[x]$ is the ring of polynomials over $\mathbb{Z}_2$. The sequence S is *periodic* iff $\exists p \in \mathbb{Z}^+ \ \forall t \in \mathbb{N}\ s_t = s_{t+p}$. The sequence S is *eventually periodic* iff $\exists \lambda \in \mathbb{N}\ \exists p \in \mathbb{Z}^+ \ \forall t \geq \lambda\ s_t = s_{t+p}$. The *generating function* of sequence S is $G(x) = \sum_{j \geq 0} s_j x^j$. The *characteristic polynomial of LSR* is $f(x) = \sum_{j=0}^{n} c_j x^j$, where for $0 \leq j < n$, $c_j \in \mathbb{Z}_2$ is the jth tap, and $c_0 = c_n = 1$. Let S_f denote the *set of nonzero output strings* generated by a LSR with characteristic polynomial f. Let sequence $S \in S_f$ and $p \in \mathbb{Z}^+$. We say *p is a period* of S iff $\forall t \in \mathbb{N}\ s_t = s_{t+p}$. Define $\text{period}(S) = \min\ \{p \in \mathbb{Z}^+ : p$ is a period of $S\}$. Let $h \in \mathbb{Z}_2[x]$ of degree d. The *reciprocal* of h is $h^*(x) = x^d h(1/x)$.

Since R has $2^n - 1$ nonzero states, the sequence S is eventually periodic with period at most $2^n - 1$. Moreover, S is periodic.[3] Because the focus of this paper is generating functions, the relationship between the generating function $G(x)$ of the sequence S and a period of S is presented in the following Lemma. This relationship is that every s_t coefficient in $G(x)$ appears exactly p positions from the previous s_t coefficient.

Lemma 1: p is a period of $S \Longleftrightarrow G(x) = \left(\sum_{j=0}^{p-1} s_j x^j\right)\left(\sum_{i=0}^{\infty} x^{ip}\right)$.

Proof: (necessity and sufficiency)

($\Longleftarrow$) Assume $G(x) = \left(\sum_{j=0}^{p-1} s_j x^j\right)\left(\sum_{i=0}^{\infty} x^{ip}\right)$. We must show that p is a period of S.

By the definition of period of S, we must show that $\forall t \in \mathcal{N}\ s_t = s_{t+p}$. By expanding the first part of $G(x)$, we see that:

$$\begin{aligned} G(x) &= (s_0 + s_1 x + \cdots + s_{p-1}x^{p-1}) \sum_{i=0}^{\infty} x^{ip} \\ &= s_0 \sum_{i=0}^{\infty} x^{ip} + s_1 x \sum_{i=0}^{\infty} x^{ip} + \cdots + s_{p-1}x^{p-1} \sum_{i=0}^{\infty} x^{ip}. \end{aligned} \tag{20}$$

By writing out the terms of $G(x)$ in order and grouping them as follows

$$\begin{aligned} G(x) &= \underbrace{s_0 + s_1 x + \cdots + s_{p-1}x^{p-1}} + \underbrace{s_0 x^p + s_1 x^{p+1} + \cdots + s_{p-1}x^{2p-1}} \\ &\quad + \underbrace{s_0 x^{2p} + s_1 x^{2p+1} + \cdots + s_{p-1}x^{3p-1}} + \cdots, \end{aligned} \tag{21}$$

[2] All definitions and lemmas come from Sherman [11].

[3] This statement can be proven, but I will not prove it here.

we see that each group has p terms. More importantly, every s_t term appears exactly p positions from the previously s_t term. In other words, $s_t = s_{t+p}$. Therefore, p is a period of S.

($\Longrightarrow$) (direct proof)

This part of the proof is similar to the first part. We apply the definition that p is a period of S iff $\forall t \in \mathbb{N}\ s_t = s_{t+p}$. We expand the right side of the inequality for $G(x)$. Because p is the period of S, every s_t term must reappear every p terms in S:

$$\begin{aligned} G(x) &= \underbrace{s_0 + s_1 x + \cdots + s_{p-1}x^{p-1}} + \underbrace{s_0 x^p + s_1 x^{p+1} + \cdots + s_{p-1}x^{2p-1}} \\ &\quad + \underbrace{s_0 x^{2p} + s_1 x^{2p+1} + \cdots + s_{p-1}x^{3p-1}} + \cdots. \end{aligned} \tag{22}$$

But Eq. 22 is exactly what we derived in Eq. 21. ∎

Lemma 1 explains how a period of S relates to the generating function $G(x)$. The next lemma takes a closer look at $G(x)$.

Lemma 2: $G(x) = \dfrac{\psi(x)}{f^*(x)}$, where $\psi(x) = \displaystyle\sum_{i=1}^{n} c_i x^{n-i} \left(\sum_{j=0}^{i-1} s_j x^j \right)$.

Proof: (definition)
Before proving the equality, we must compute $f^*(x)$:

$$\begin{aligned} f^*(x) &= x^n f(\frac{1}{x}) \\ &= x^n \sum_{i=0}^{n} c_i \left(\frac{1}{x}\right)^i \\ &= \sum_{i=0}^{n} c_i x^{n-i}. \end{aligned} \tag{23}$$

We know the value for $G(x)$:

$$G(x) = \sum_{t=0}^{\infty} s_t x^t = \sum_{t=0}^{n-1} s_t x^t + \sum_{t=0}^{\infty} s_{t+n} x^{t+n}. \tag{24}$$

We also know the definition of the feedback function of the LSR:

$$s_{t+n} = \sum_{i=0}^{n-1} c_i s_{t+i}. \tag{25}$$

In order to prove the equality we can rearrange the equation and check that $G(x)f^*(x) = \psi(x)$. To do this we substitute Eq. 25 into Eq. 24:

$$G(x) = \sum_{t=0}^{n-1} s_t x^t + \sum_{t=0}^{\infty} \left(\sum_{i=0}^{n-1} c_i s_{t+i} \right) x^{t+n}$$

$$\begin{aligned} &= \sum_{t=0}^{n-1} s_t x^t + \sum_{i=0}^{n-1} \left(c_i x^{n-i} \left(\sum_{t=0}^{\infty} s_{t+i} x^{t+i} \right) \right) \\ &= \sum_{t=0}^{n-1} s_t x^t + c_0 x^n \sum_{t=0}^{\infty} s_t x^t + \sum_{i=1}^{n-1} \left(c_i x^{n-i} \left(\sum_{t=0}^{\infty} s_t x^t + \sum_{j=0}^{i-1} s_j x^j \right) \right) \\ &= \sum_{t=0}^{n-1} s_t x^t + c_0 x^n G(x) + \sum_{i=1}^{n-1} \left(c_i x^{n-i} \left(G(x) + \sum_{j=0}^{i-1} s_j x^j \right) \right). \end{aligned} \tag{26}$$

Because addition over $\mathbb{Z}_2$ implies that "+" and "−" are the same, we use "+" in the last two lines of the above derivation for Eq. 26. Now we can rearrange Eq. 26 such that

$$G(x) \left(1 + \sum_{i=1}^{n-1} c_i x^{n-i} + c_0 x^n \right) = \sum_{t=0}^{n-1} s_t x^t + \sum_{i=1}^{n-1} c_i x^{n-i} \left(\sum_{j=0}^{i-1} s_j x^j \right). \tag{27}$$

We simplify Eq. 27 to become

$$G(x) f^*(x) = \sum_{i=1}^{n} c_i x^{n-i} \left(\sum_{j=0}^{i-1} s_j x^j \right), \text{where } c_n = 1. \tag{28}$$

Therefore we have proven that $G(x)f^*(x) = \psi(x)$. ■

Lemma 2 shows explicitly how the sequence S depends only on $f(x)$ and the initial fill x_0. This lemma essentially solves the recurrence.

6 Probability Generating Functions

Probability generating functions are not used to solve recurrences, but they are important because they are used in computing moments of probability distributions. The derivatives of the probability generating functions are used to calculate these moments. We show how to calculate the mean and variance, and then present the queuing network model as an application of probability generating functions.

Let X be a random variable that takes only positive integer values. We can define a probability generating function (pgf) of X as

$$G(z) = \sum_{n=0}^{\infty} \mathrm{P}(X = k) z^k. \tag{29}$$

The coefficients of $G(z)$ are positive and sum to 1. This means that $G(1) = 1$. It is not difficult to calculate the mean and variance of the pgf G above. Below we show how the mean and variance of G are derived:

Mean: The mean is given by:

$$\begin{aligned} E[X] &= \sum_{k \geq 0} k \cdot \mathrm{P}(X = k) \\ &= \sum_{k \geq 0} \mathrm{P}(X = k) \cdot k z^{k-1} |_{z=1} \end{aligned}$$

Distribution	Formula for p_k	PGF
Binomial	$\binom{n}{k} p^k(1-p)^{n-k}$	$(1-p+pz)^n$
Geometric	pq^k	$\frac{1-q}{1-qz}$
Poisson	$\lambda^k e^{-\lambda}/k!$	$e^{\lambda(z-1)}$
Exponential	$\lambda e^{-\lambda k}$	$\frac{\lambda}{k+\lambda}$

Table 3: Probability Generating Functions for Some Random Variables

We differentiate the pgf with respect to z and set $z = 1$ to derive

$$\text{Mean}(G) = G^{'}(1). \tag{30}$$

Variance: In order to calculate the variance we first need to calculate $E[X^2]$:

$$\begin{aligned} E[X^2] &= \sum_{k\geq 0} k^2 \cdot \text{P}(X=k) \\ &= \sum_{k\geq 0} \text{P}(X=k) \cdot (k(k-1)z^{k-2} + kz^{k-1})|_{z=1} \\ &= G^{''}(1) + G^{'}(1). \end{aligned} \tag{31}$$

Variance is equal to $E[X^2] - (E[X])^2$, so we use Eq. 17 to compute

$$\text{Var}(G) = G^{''}(1) + G^{'}(1) - G^{'}(1)^2. \tag{32}$$

Although Eqs. 30 and 32 reveal only the first moment, higher moments can be calculated. For details on these calculations and other pgf properties, see Graham [3]. There are several probability distributions, each with their own pgf, mean, and variance. Table 3 shows the probability generating functions for four commonly encountered distributions.

What is the probability that a given event in a specified protocol will occur? Once we determine this probability, we can determine how a system will operate for a given event. The queuing network model is an example of such a system. In this model, an event is the arrival of a message/job that is to be placed in the queue. Below, we describe the setup of the queuing network model and calculate the probability of n arrivals in time t. The Poisson distribution is the basis for all computations.

Example: Queuing Network Model

An operating system may use the queuing network model in which several queues are used to process jobs. We are interested in the case where a single queue processes jobs. We want to know the probability that there are n arrivals in time t, given that λ is the number of messages/jobs per unit time. Define $P_n(t)$ to be the probability of n arrivals in time t. We assume that:

1) Probability of one arrival in time t is λt.

2) Probability of no arrival in time t is $(1-\lambda t)$.

3) Arrivals are independent.

To calculate $P_n(t)$ we have to set up the problem. Look at the problem with regard to the time $t+\Delta t$. We calculate the probability of no arrivals as well as the probability of n arrivals in time $t+\Delta t$:

$$P_0(t+\Delta t) = P_0(t)P_0(\Delta t) = P_0(t)(1-\lambda\Delta t). \tag{33}$$

In order to calculate the $P_n(t+\Delta t)$, we use divide and conquer to separate the problem to when we have n arrivals in time t and none in time Δt, or we have $n-1$ arrivals in time t and 1 in time Δt:

$$\begin{aligned} P_n(t+\Delta t) &= P_n(t)P_0(\Delta t) + P_{n-1}(t)P_1(\Delta t) \\ &= P_n(t)(1-\lambda\Delta t) + P_{n-1}(t)\lambda\Delta t. \end{aligned} \tag{34}$$

Now we use Eqs. 33 and 34 in order to calculate the derivatives of $P_0(t)$ and $P_n(t)$. By taking the limit as Δt approaches 0, we calculate $P_0'(t)$ and $P_n'(t)$ to be:

$$P_0'(t) = \frac{P_0(t+\Delta t) - P_0(t)}{\Delta t} = -\lambda P_0(t) \tag{35}$$

$$P_n'(t) = \frac{P_n(t+\Delta t) + P_n(t)}{\Delta t} = -\lambda P_n(t) + \lambda P_{n-1}(t). \tag{36}$$

Now that we have formulas for the probabilities, we must perform steps similar to those given in Section 3 for solving recurrences.

Step 1:
We need to choose the general form of the probability generating function. Let $G(z,t)$ denote the pgf. There are two parameters since G is also a function of time t. Let $G(z,t) = \sum_{n\geq 0} P_n(t)z^n$. Define $P_n(0)$, the probability of n arrivals in 0 time, as

$$P_n(0) = \begin{cases} 0 & n \neq 0 \\ 1 & n = 0, \end{cases} \tag{37}$$

We now compute two trivial cases for $G(z,t)$. The first case sets time to 0, and the second case sets z to 1:

$$G(z,0) = \sum_{n\geq 0} P_n(0)z^n = P_0(0) = 1, \tag{38}$$

$$G(1,t) = \sum_{n\geq 0} P_n(t)1^n = P_n(1) = 1. \tag{39}$$

Notice that all terms of the summation of Eq. 38 are zero except for $P_0(0)$, and that Eq. 39 is exactly what was mentioned earlier (*i.e.* when $z = 1$, $G(z,t) = 1$ for any time t). Now we state formulas for $P_0(t)$ and $P_n(t)$:

$$\begin{aligned} P_0(t) &= G(z,t)|_{z=0} && (40)\\ P_n(t) &= \frac{1}{n!}\left(\frac{d}{dz}\right)^n G(z,t)|_{z=0} && (41) \end{aligned}$$

Step 2:
We derive a formula for $G(z,t)$ by using the definition of $G(z,t)$. We start by taking the derivative of both sides of the formula:

$$\frac{d}{dt}G(z,t) = \frac{d}{dt}\sum_{n\geq 0} P_n(t)z^n = \sum_{n\geq 0} P_n'(t)z^n. \tag{42}$$

Using Eq. 36 for $P_n'(t)$, Eq. 42 becomes

$$\begin{aligned} \frac{d}{dt}G(z,t) &= \sum_{n\geq 0}\left(-\lambda P_n(t) + \lambda P_{n-1}(t)\right) z^n \\ &= -\lambda \sum_{n\geq 0} P_n(t)z^n + \lambda z \sum_{n\geq 0} P_n(t)z^n \\ &= -\lambda G(z,t) + \lambda z G(z,t) \\ &= \lambda(z-1)G(z,t). && (43) \end{aligned}$$

By integrating both sides of Eq. 43, we obtain

$$G(z,t) = ce^{\lambda(z-1)t}. \tag{44}$$

To calculate c, we set $t = 0$ and Eq. 44 equal to the result obtained from Eq. 38 above:

$$G(z,0) = ce^0 = c = 1. \tag{45}$$

Step 3:
We solve for $P_n(t)$ by substituting for $G(z,t)$ in Eq. 41 with Eq. 44 where $c = 1$:

$$\begin{aligned} P_n(t) &= \frac{1}{n!}\left(\frac{d}{dz}\right)^n e^{\lambda(z-1)t}|_{z=0} \\ &= \frac{e^{-\lambda t}(\lambda t)^n e^{\lambda t z}}{n!}|_{z=0} \\ &= \frac{(\lambda t)^n e^{-\lambda t}}{n!}. && (46) \end{aligned}$$

Other examples of applications using probability generating functions are presented in Lueker [8]. For more general information on pgf's, see Graham [4].

7 Conclusion

We have discussed generating functions and the method used in solving recurrences. Because of the sequential nature of recurrences, generating functions apply. We applied the three-step method in order to solve the recurrence for the Fibonacci numbers with generating functions. We have shown how generating functions apply to any situation dealing with sequences, especially with the sequence derived by linear shift registers. We have presented probability generating functions and applied them to the queuing network model. Generating functions are powerful tools used in transforming a problem from one domain into another domain in which the problem is easier to solve and whose solution can be justified.

Several combinatorial texts have a chapter devoted to generating functions. Roberts [10] and Stanley [13] specifically have a chapter on generating functions and recurrences. The basic operations of generating functions are provided by Hillman [5] and Knuth [6]. For more examples using generating functions, see Bender [1] and Cohen [3]. More detailed information on generating functions and probability generating functions is presented by Graham [4]. For a text completely devoted to generating functions, I recommend Wilf [13]. Generating functions are not only used to solve recurrences; they are often used in solving partition problems. For the interested reader, readings on how generating functions are used in partition problems are found in Butler [2] and Okada [9]. The reader is encouraged to consult these references.

Acknowledgments

I am thankful for the helpful comments and suggestions of Alan Sherman and Richard Chang.

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APPENDIX

Exercises on Generating Functions:

1) Find the generating function for the sequence $(0, 0, 4, 4, 4, \ldots)$ using the differencing method. Explain your answer.

2) Find the generating function for the sequence $(0, 0, 0, 3, 6, 12, 24, \ldots)$ using the differencing method. Explain your answer.

3) Find the sequence whose generating function is $(x + 6)^2$.

4) Find the sequence whose generating function is $x^3 + x^4 + e^{2x}$.

5) Find the sequence whose generating function is $3/(1 + 4x) - 2/(1 - 5x)$.

6) Find the generating function for the recurrence $a_n = 10a_{n-1} - 16a_{n-2}$ where $a_0 = 0$, $a_1 = 1$. Find the nth coefficient. Check your answer using characteristic equations.

7) Find the generating function for the recurrence $b_n = 6b_{n-1} - 11b_{n-2} + 6b_{n-3}$ where $b_0 = 0, b_1 = 1, b_2 = 2$. Find the nth coefficient. Check your answer using characteristic equations.

8) Find the generating function for the recurrence $t_n = 2t_{n-1} + n + 2^n$ where $t_0 = 0$. Find the nth coefficient.

9) Find the generating function for the recurrence $s_n = s_{n-1} + n^2$ where $s_0 = 0$. Find the nth coefficient.

10) Find the generating functions and the nth coefficients for the nested recurrences

$$\begin{aligned} a_n &= a_{n-1} + b_{n-1} + c_{n-1} \\ b_n &= 4^n - c_{n-1} \\ c_n &= 4^n - b_{n-1} \end{aligned}$$

with $n \geq 1$ and $a_0 = b_0 = c_0 = 1$.

Solutions to Exercises:

Problem 1.
The sequence is $(0, 0, 4, 4, 4, \ldots)$. The power series is $G(x) = 4x^2 + 4x^3 + 4x^4 + \cdots$. Calculate $G(x)$ multiplied by $-x$, and add the result to $G(x)$ to solve for $G(x)$: equations we get these equations

$$\begin{aligned} G(x) &= 4x^2 + 4x^3 + 4x^4 + \cdots \\ -xG(x) &= -4x^3 - 4x^4 - \cdots \\ (1-x)G(x) &= 4x^2. \end{aligned}$$

The closed form of $G(x)$ is $4x^2(1-x)^{-1}$.

This answer makes sense by comparing the sequence $(0, 0, 4, 4, 4, \ldots)$ with the sequence $(1, 1, 1, \ldots)$ whose generating function is $(1-x)^{-1}$. Now look at the sequence $(0, 0, 1, 1, 1, \ldots)$ which is the result of shifting $(1, 1, 1, \ldots)$ two positions to the right. This is achieved by the generating function $x^2(1-x)^{-1}$. Multiplying this sequence by 4 gives $(0, 0, 4, 4, 4, \ldots)$ so the final generating function is $4x^2(1-x)^{-1}$.

Problem 2.
The sequence is $(0, 0, 0, 3, 6, 12, 24, \ldots)$. The power series is $G(x) = 3x^3 + 6x^4 + 12x^5 + 24x^6 + \cdots$. Calculate $G(x)$ multiplied by $-2x$, and add the result to $G(x)$ to solve for $G(x)$:

$$\begin{aligned} G(x) &= 3x^3 + 6x^4 + 12x^5 + 24x^6 + \cdots \\ -2xG(x) &= -6x^4 - 12x^5 - 24x^6 - \cdots \\ (1-2x)G(x) &= 3x^3. \end{aligned}$$

The closed form of $G(x)$ is $3x^3(1-2x)^{-1}$.

This answer makes sense by comparing the sequence $(0, 0, 0, 3, 6, 12, 24, \ldots)$ with the sequence $(1, 2, 4, 8, \ldots)$ whose generating function is $(1-2x)^{-1}$. By multiplying the sequence by x^3, we shift the sequence three positions to the right to get $(0, 0, 0, 1, 2, 4, 8, \ldots)$. Now multiply by 3 to get the sequence $(0, 0, 0, 3, 6, 12, 24, \ldots)$ and the resulting generating function is $3x^3(1-2x)^{-1}$.

Problem 3.
Use the binomial theorem: $(x+y)^n = \sum_{j=0}^{n} \begin{pmatrix} n \\ j \end{pmatrix} x^j y^{n-j}$.
Let $y = 6$ and $n = 2$ to get: $(x+6)^2 = \sum_{j=0}^{2} \begin{pmatrix} 2 \\ j \end{pmatrix} x^j 6^{2-j}$.
The result of the expanding the summation and simplifying the combination values is

$$(x+6)^2 = 1 \cdot 6^2 + 2 \cdot 6x + 1 \cdot x^2 = 36 + 12x + x^2.$$

The sequence is therefore $(36, 12, 1, 0, 0, \ldots)$.

Problem 4.
We need to use the Maclaurin expansion for e^x:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.$$

Substitute $2x$ for x to get

$$e^{2x} = \sum_{k=0}^{\infty} \frac{(2x)^k}{k!} = 1 + 2x + \frac{4}{2!}x^2 + \frac{8}{3!}x^3 + \frac{16}{4!}x^4 + \frac{32}{5!}x^5 + \cdots.$$

When we add $x^3 + x^4$ to the result of e^{2x} and combine terms, we get

$$1 + 2x + \frac{4}{2!}x^2 + \left(\frac{8}{3!} + 1\right)x^3 + \left(\frac{16}{4!} + 1\right)x^4 + \frac{32}{5!}x^5 + \cdots.$$

This gives the sequence

$$\left(1, 2, \frac{4}{2!}, \frac{8}{3!} + 1, \frac{16}{4!} + 1, \frac{32}{5!}, \frac{64}{6!}, \ldots\right).$$

Problem 5.
Finding x_n for all $n \in \mathcal{N}$ will suffice. Split the generating function into two parts: $3/(1+4x)$ and $2/(1-5x)$. We use the second generating function of Table 2 for the sequence $(1, c, c^2, \ldots)$. If $c = -4$ and we multiply the generating funcion by 3, then the nth term in the sequence if $3(-4)^n$. If $c = 5$ and we multiply the generating function by 2, then the nth term in the sequence is $2(5)^n$. Combining these results,the sequence for $3/(1+4x) - 2/(1-5x)$ is given by $(x_0, x_1, \ldots, x_n, \ldots)$ where $x_n = 3(-4)^n - 2(5)^n$.

Problem 6.

The recurrence is $a_n = 10a_{n-1} - 16a_{n-2}$ and $a_0 = 0, a_1 = 1$. We start by multiplying by x^n and summing over all n:

$$\sum_{n=2}^{\infty} a_n x^n = 10 \sum_{n=2}^{\infty} a_{n-1} x^n - 16 \sum_{n=2}^{\infty} a_{n-2} x^n.$$

The left side of the equation is the same as

$$a_0 + a_1 x + \sum_{n=2}^{\infty} a_n x^n - a_0 - a_1 x = \sum_{n=0}^{\infty} a_n x^n - a_0 - a_1 x.$$

The first part of the right side of the equation is the same as

$$10x \sum_{n=1}^{\infty} a_n x^n = 10x \left(a_0 + \sum_{n=1}^{\infty} a_n x^n - a_0 \right) = 10x \sum_{n=0}^{\infty} a_n x^n - 10x a_0.$$

The second part of the right side of the equation is the same as $16x^2 \sum_{n=0}^{\infty} a_n x^n$.
Substituting $G(x)$ for $\sum_{n=0}^{\infty} a_n x^n$ in the above results and using $a_0 = 0, a_1 = 1$ we obtain

$$G(x) - x = 10xG(x) - 16x^2 G(x).$$

We solve for $G(x)$:

$$G(x) = \frac{x}{1 - 10x + 16x^2} = \frac{x}{(1-8x)(1-2x)}.$$

We expand $G(x)$ into a power series to compute a_n. Using the method of partial fraction decomposition,

$$G(x) = \frac{1}{6}(1-8x)^{-1} - \frac{1}{6}(1-2x)^{-1}.$$

Therefore $a_n = 8^n/6 - 2^n/6$.

To check the answer by characteristic equations we must rewrite the recurrence as $a_n - 10a_{n-1} + 16a_{n-2} = 0$. The characteristic equation is $r^2 - 10r + 16 = 0$ from which we compute roots $r_1 = 2$ and $r_2 = 8$. Now we need to find the constant terms, so we state the equations using the initial conditions:

$$\begin{aligned} c_1 + c_2 &= 0 \\ 2c_1 + 8c_2 &= 1. \end{aligned}$$

When we solve the two equations, we calculate $c_1 = -1/6$ and $c_2 = 1/6$. So $a_n = 8^n/6 - 2^n/6$.

Problem 7.

The recurrence is $b_n = 6b_{n-1} - 11b_{n-2} + 6b_{n-3}$ and $b_0 = 0, b_1 = 1, b_2 = 2$. We multiply by x^n and sum over all n:

$$\sum_{n=3}^{\infty} b_n x^n = 6\sum_{n=3}^{\infty} b_{n-1}x^n - 11\sum_{n=3}^{\infty} b_{n-2}x^n + 6\sum_{n=3}^{\infty} b_{n-3}x^n.$$

After manipulating the summations as in problem 6 we obtain

$$\sum_{n=0}^{\infty} b_n x^n - b_0 - b_1 x - b_2 x^2 = 6x\left(\sum_{n=0}^{\infty} b_n x^n - b_0 - b_1 x\right) - 11x^2\left(\sum_{n=0}^{\infty} b_n x^n - b_0\right) + 6x^3\sum_{n=0}^{\infty} b_n x^n.$$

Substituting $G(x)$ for $\sum_{n=0}^{\infty} b_n x^n$ and using the initial conditions we compute

$$G(x) - x - 2x^2 = 6xG(x) - 6x^2 - 11x^2G(x) + 6x^3G(x).$$

Solving for $G(x)$ we get

$$G(x) = \frac{x - 4x^2}{1 - 6x + 11x^2 - 6x^3} = \frac{x - 4x^2}{(1-x)(1-2x)(1-3x)}.$$

To get b_n we expand $G(x)$ into a power series. Using the method of partial fraction decomposition, we get

$$G(x) = -\frac{3}{2}(1-x)^{-1} + 2(1-2x)^{-1} - \frac{1}{2}(1-3x)^{-1}.$$

Therefore ,

$$b_n = -\frac{3}{2}(1)^n + 2(2)^n - \frac{1}{2}(3)^n.$$

To check the answer by characteristic equations we must rewrite the recurrence as $b_n - 6b_{n-1} + 11b_{n-2} - 6b_{n-3} = 0$. The characteristic equation is $r^3 - 6r^2 + 11r - 6 = 0$ whose roots are $r_1 = 1, r_2 = 2$ and $r_3 = 3$. Now we need to find the constant terms so we state the equations using the initial conditions:

$$\begin{aligned} 0 &= c_1 + c_2 + c_3 \\ 1 &= c_1 + 2c_2 + 3c_3 \\ 2 &= c_1 + 4c_2 + 9c_3. \end{aligned}$$

When we solve the three equations we calculate $c_1 = -3/2, c_2 = 2$, and $c_3 = -1/2$. Therefore,

$$b_n = -\frac{3}{2}(1)^n + 2(2)^n - \frac{1}{2}(3)^n.$$

Problem 8.

The recurrence is $t_n = 2t_{n-1} + n + 2^n$ and $t_0 = 0$. We multiply by x^n and sum over all n:

$$\sum_{n=1}^{\infty} t_n x^n = 2\sum_{n=1}^{\infty} t_{n-1} x^n + \sum_{n=1}^{\infty} n x^n + \sum_{n=1}^{\infty} 2^n x^n.$$

After manipulating the summations we obtain

$$\sum_{n=0}^{\infty} t_n x^n - t_0 = 2x\sum_{n=0}^{\infty} t_n x^n + \sum_{n=0}^{\infty} n x^n + \sum_{n=0}^{\infty} 2^n x^n - 2^0.$$

We need to find the generating function closed form for $\sum_{n=0}^{\infty} n x^n$.
We know that

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} n x^{n-1}.$$

If we multiply this formula by x we derive the summation we need. We use $G(x) = \sum_{n=0}^{\infty} t_n x^n$, the initial conditions, and the closed forms for the last two summations:

$$G(x) = 2xG(x) + \frac{x}{(1-x)^2} + \frac{1}{1-2x} - 1.$$

We solve for $G(x)$:

$$\begin{aligned}
G(x) &= \frac{1}{1-2x}\left[\frac{x}{(1-2x)^2} + \frac{1}{1-2x} - 1\right] \\
&= \frac{x}{(1-2x)(1-x)^2} + \frac{1}{(1-2x)^2} - \frac{1}{1-2x} \\
&= \frac{2x^3 - 6x^2 + 3x}{(1-x)^2(1-2x)^2} \\
&= \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{C}{1-2x} + \frac{D}{(1-2x)^2}.
\end{aligned}$$

We calculate $A = -1, B = -1, C = 1, D = 1$, therefore

$$\frac{2x^3 - 6x^2 + 3x}{(1-x)^2(1-2x)^2} = \frac{-1}{1-x} + \frac{-1}{(1-x)^2} + \frac{1}{1-2x} + \frac{1}{(1-2x)^2}.$$

In order to find the nth coefficient we need the nth coefficient for each term in the equation. The first term gives us -1 since the generating function gives a coefficient of 1 and we multiply this by -1. The third term gives us 2^n. We have to use the binomial theorem for the second and fourth terms:

$$\begin{aligned}(-1)(1-x)^{-2} &= (-1)\sum_{n=0}^{\infty}\binom{-2}{n}(-x)^n \\ (1-2x)^{-2} &= \sum_{n=0}^{\infty}\binom{-2}{n}(-2x)^n.\end{aligned}$$

From the equation for $(-1)(1-x)^{-2}$ we see that the nth coefficient is

$$(-1)\binom{-2}{n}(-1)^n = (-1)(-1)^n\binom{2+n-1}{n}(-1)^n = (-1)\binom{n+1}{n} = (-1)(n+1) = -n-1.$$

From the equation for $(1-2x)^{-2}$ we see that the nth coefficient is

$$\binom{-2}{n}(-2)^n = (-1)^n\binom{2+n-1}{n}(-2)^n = 2^n\binom{n+1}{n} = 2^n(n+1) = n2^n + 2^n.$$

We sum the nth coefficient values for each term to compute

$$\begin{aligned}x_n &= -1 + 2^n - n - 1 + n2^n + 2^n \\ &= -2 - n + 2(2)^n + n(2)^n.\end{aligned}$$

Problem 9.

The recurrence is $s_n = s_{n-1} + n^2$ and $s_0 = 0$. We multiply by x^n and summing over all n:

$$\sum_{n=1}^{\infty} s_n x^n = \sum_{n=1}^{\infty} s_{n-1}x^n + \sum_{n=1}^{\infty} n^2 x^n.$$

After maniplulating the summations,

$$\sum_{n=0}^{\infty} s_n x^n - s_0 = x\sum_{n=0}^{\infty} s_n x^n + \sum_{n=0}^{\infty} n^2 x^n.$$

We need to find the generating function closed form for the last summation. From problem 8 we know that $\sum_{n=0}^{\infty} nx^n$ has the closed form of $x/(1-x)^2$. If we take the derivative of this generating function we compute

$$\frac{d}{dx}\left[\frac{x}{(1-x)^2}\right] = \frac{x+1}{(1-x)^3} = \sum_{n=0}^{\infty} n^2 x^{n-1}$$

Now we multiply this answer by x:

$$\frac{x(x+1)}{(1-x)^3} = \sum_{n=0}^{\infty} n^2 x^n.$$

We use $G(x) = \sum_{n=0}^{\infty} s_n x^n$, our initial conditions, and the closed form that we just derived for the last summation to compute

$$G(x) = xG(x) + \frac{x+x^2}{(1-x)^3}.$$

We solve for $G(x)$:

$$G(x) = \frac{x+x^2}{(1-x)^4} = \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{C}{(-1x)^3} + \frac{D}{(1-x)^4}.$$

We calculate $A = 0, B = 1, C = -3, D = 2$, therefore

$$\frac{x+x^2}{(1-x)^4} = \frac{1}{(1-x)^2} - \frac{3}{(1-x)^3} + \frac{2}{(1-x)^4}.$$

If we use the binomial theorem we see that the nth coefficient of the first term is

$$\binom{-2}{n}(-1)^n = (-1)^n \binom{2+n-1}{n}(-1)^n = \binom{n+1}{n} = n+1.$$

From the second term we calculate

$$-3\binom{-3}{n}(-1)^n = -3(-1)^n \binom{3+n-1}{n}(-1)^n = -3\binom{n+2}{n} = -\frac{3}{2}(n+1)(n+2).$$

From the third term we calculate

$$2\binom{-4}{n}(-1)^n = 2(-1)^n \binom{4+n-1}{n}(-1)^n = 2\binom{n+3}{n} = \frac{1}{3}(n+3)(n+2)(n+1).$$

When we add up nth coefficient values for each term we compute

$$\begin{aligned} x_n &= (n+1) - \frac{3}{2}(n+1)(n+2) + \frac{1}{3}(n+3)(n+2)(n+1) \\ &= \frac{1}{6}\left[(n+1)(2n^2+n)\right] \\ &= \frac{1}{6}(n)(n+1)(2n+1). \end{aligned}$$

Problem 10.

To solve the nested recurrences, we use the same steps as with any recurrences. We multiply both sides of each of our equations by x^n and sum over all n:

$$\begin{aligned}
\sum_{n=1}^{\infty} a_n x^n &= \sum_{n=1}^{\infty} a_{n-1} x^n + \sum_{n=1}^{\infty} b_{n-1} x^n + \sum_{n=1}^{\infty} c_{n-1} x^n, \\
\sum_{n=1}^{\infty} b_n x^n &= \sum_{n=1}^{\infty} 4^n x^n - \sum_{n=1}^{\infty} c_{n-1} x^n, \\
\sum_{n=1}^{\infty} c_n x^n &= \sum_{n=1}^{\infty} 4^n x^n - \sum_{n=1}^{\infty} b_{n-1} x^n.
\end{aligned}$$

If we use as our generating functions

$$A(x) = \sum_{n=0}^{\infty} a_n x^n, B(x) = \sum_{n=0}^{\infty} b_n x^n, C(x) = \sum_{n=0}^{\infty} c_n x^n$$

we calculate

$$\begin{aligned}
A(x) - a_0 &= xA(x) + xB(x) + xC(x), \\
B(x) - b_0 &= \frac{1}{1-4x} - 4^0 - xC(x), \\
C(x) - c_0 &= \frac{1}{1-4x} - 4^0 - xB(x).
\end{aligned}$$

Using the initial conditions we see that

$$\begin{aligned}
A(x) &= \frac{1}{1-x}\left[xB(x) + xC(x) + 1\right], \\
B(x) &= \frac{1}{1-4x} - xC(x), \\
C(x) &= \frac{1}{1-4x} - xB(x).
\end{aligned}$$

It is easy to see that $B(x) = C(x)$. By substituting $B(x)$ for $C(x)$ in the second equation we compute

$$B(x) = C(x) = \frac{1}{(1-4x)(1+x)} = \frac{4}{5}(1-4x)^{-1} + \frac{1}{5}(1+x)^{-1}.$$

Therefore,

$$b_n = c_n = \frac{4}{5}(4)^n + \frac{1}{5}(-1)^n.$$

By substituting for $B(x)$ and $C(x)$ in the first equation we compute

$$A(x) = \frac{1 - x - 4x^2}{(1-x)(1-4x)(1+x)} = \frac{2}{3}(1-x)^{-1} + \frac{8}{15}(1-4x)^{-1} - \frac{1}{5}(1+x)^{-1}.$$

Therefore,

$$a_n = \frac{2}{3}(1)^n + \frac{8}{15}(4)^n - \frac{1}{5}(-1)^n.$$