CMSC 491/691

Lecture 11

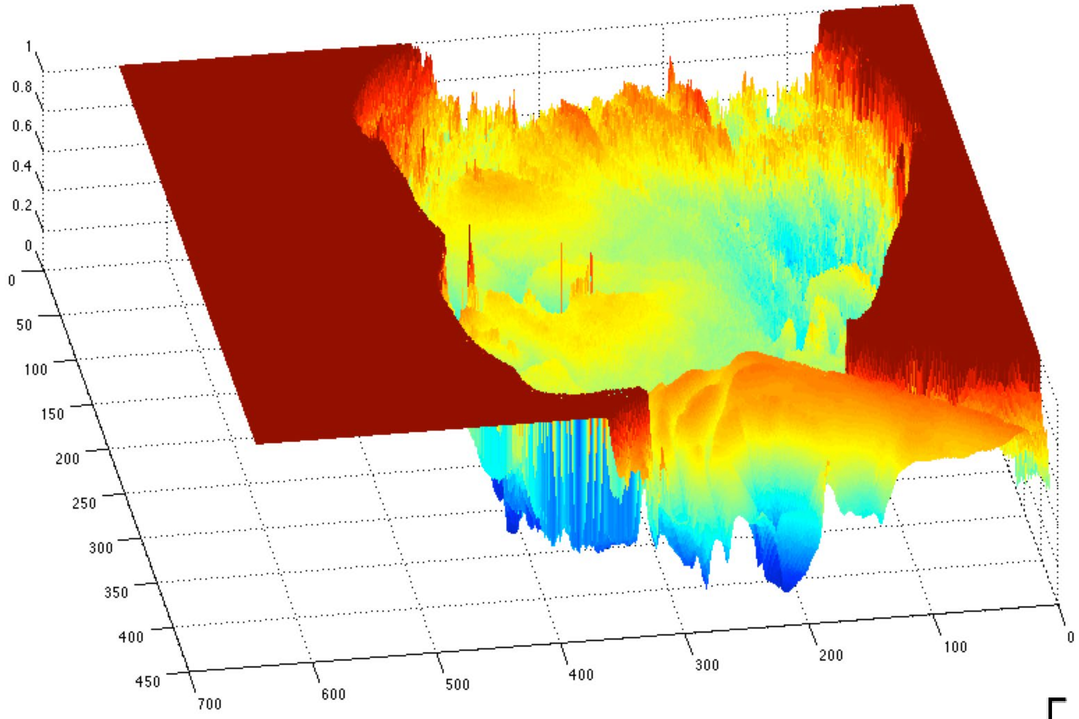
Image Transformations

Some slides from
Yang, Jayasuriya

What is an image?

grayscale image

$f(\boldsymbol{x})$



A (grayscale) image is a 2D function.

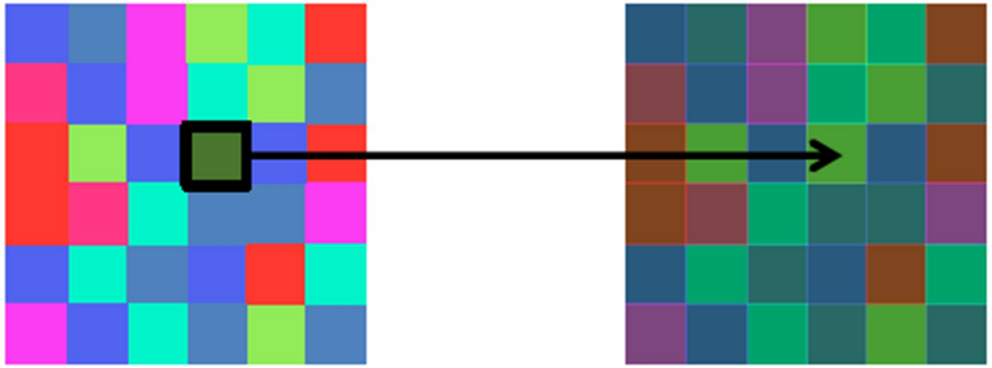
What is the range of the image function f?

domain $\boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix}$

RECALL

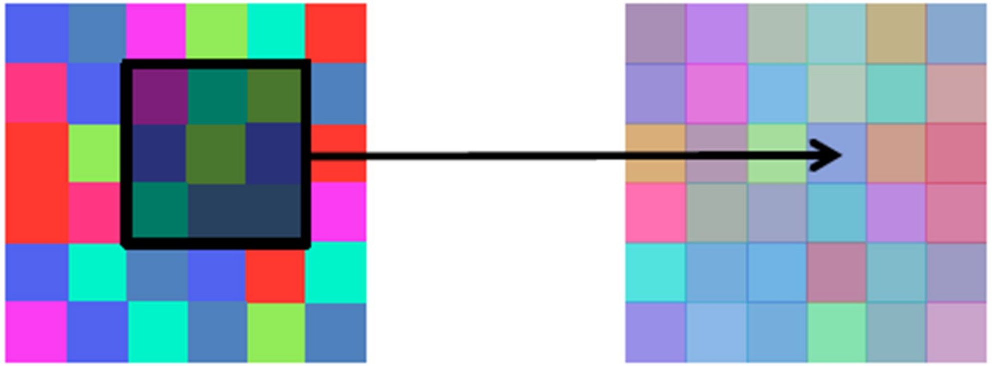
Point Processing and Image Filtering

Point Operation



point processing

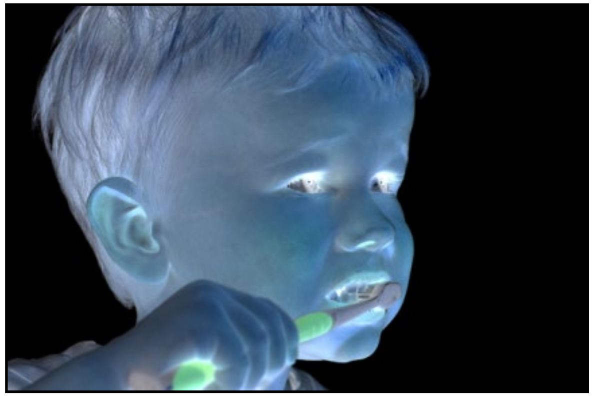
Neighborhood Operation



“filtering”

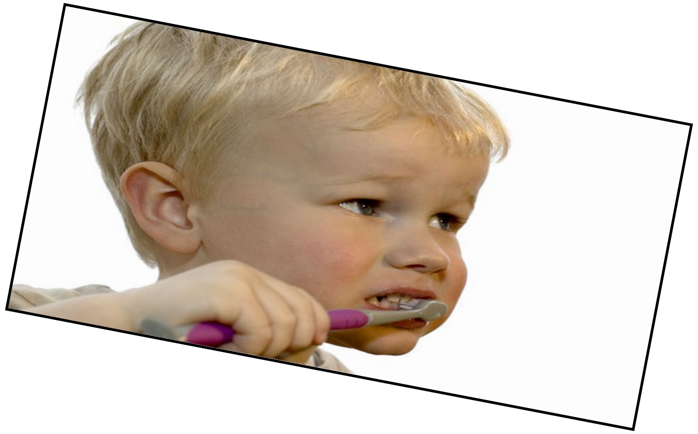
What types of image transformations can we do?

Filtering



changes pixel *values*

Warping



changes pixel *locations*

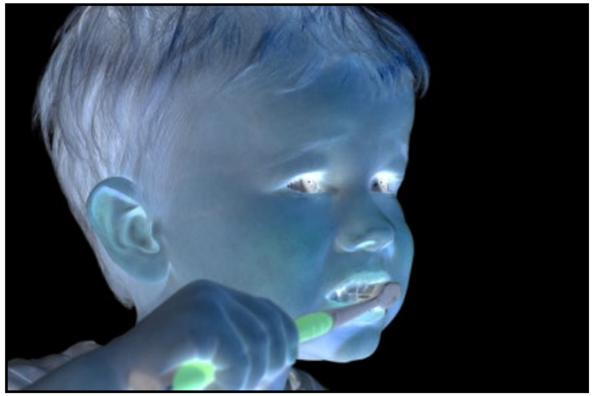
What types of image transformations can we do?

F

Filtering

$$G(\boldsymbol{x}) = h\{F(\boldsymbol{x})\}$$

G



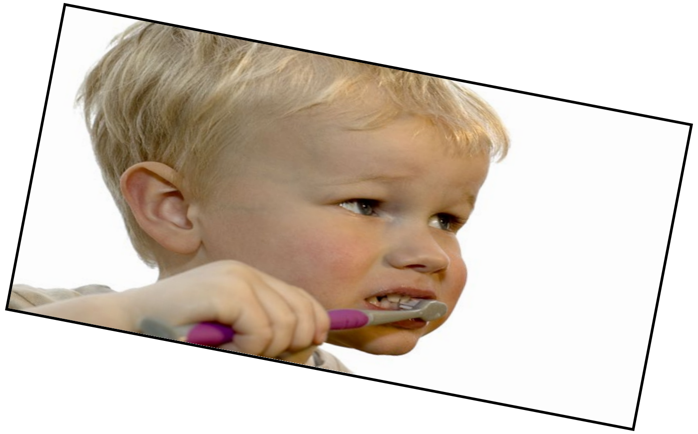
changes *range* of image function

F

Warping

$$G(\boldsymbol{x}) = F(h\{\boldsymbol{x}\})$$

G

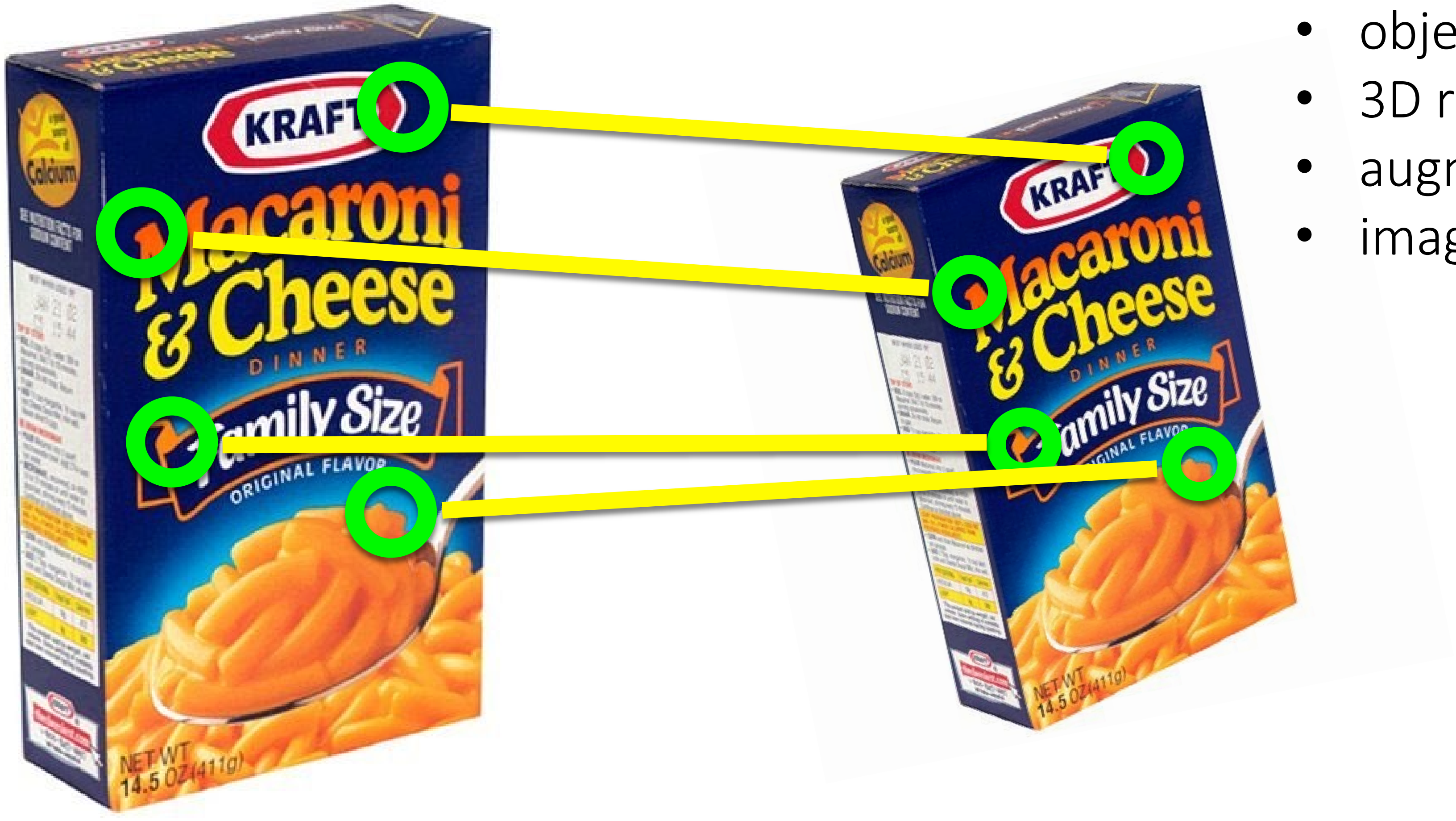


changes *domain* of image function

Warping example: feature matching

Warping example: feature matching

Warping example: feature matching



- object recognition
- 3D reconstruction
- augmented reality
- image stitching

How do you compute the transformation?

Warping example: feature matching

Given a set of matched feature points:

$$\{\boldsymbol{x}_i, \boldsymbol{x}'_i\}$$

point in one image

point in the other image

and a transformation:

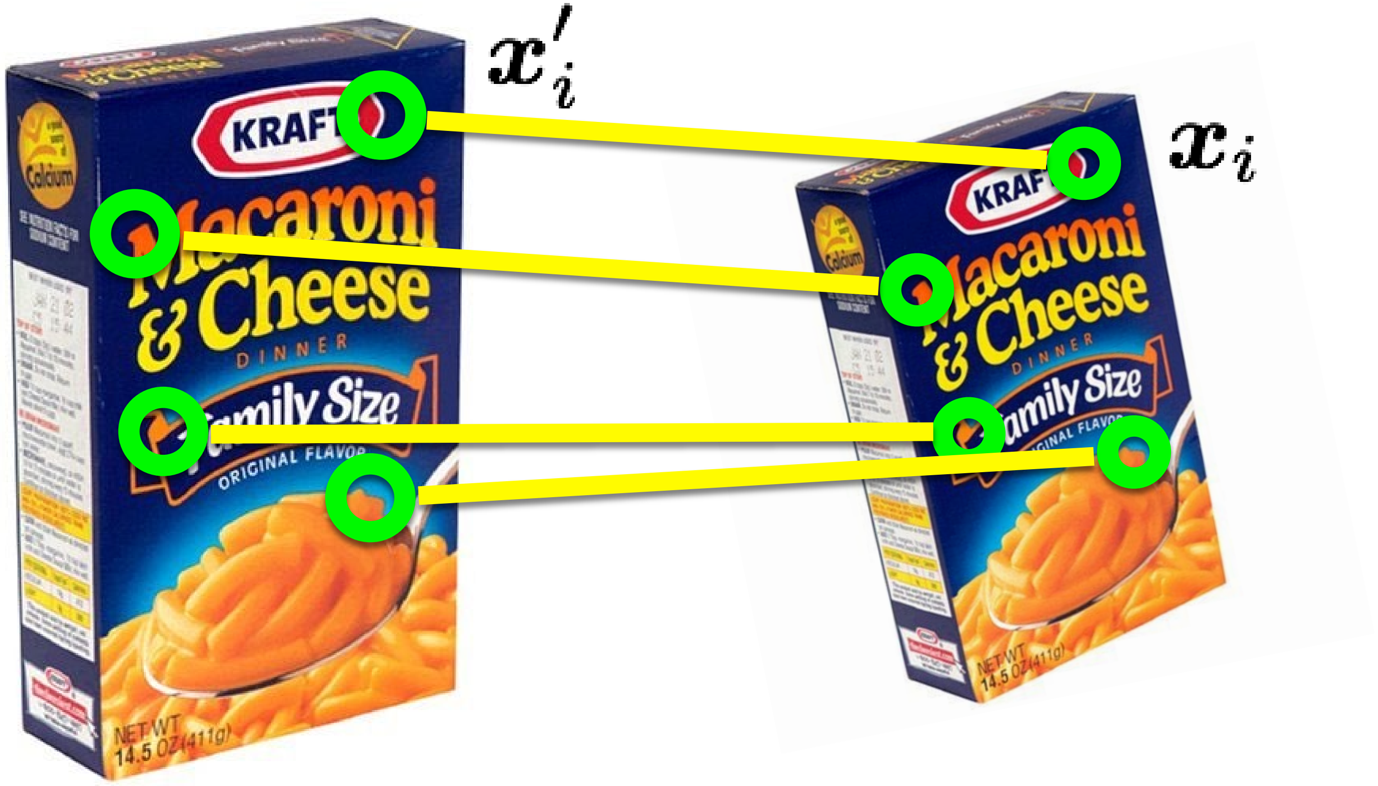
$$\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x}; \boldsymbol{p})$$

transformation function

parameters

find the best estimate of the parameters

$$\boldsymbol{p}$$



What kind of transformation functions $\boldsymbol{f}$ are there?

2D transformations

2D transformations

translation

rotation

aspect

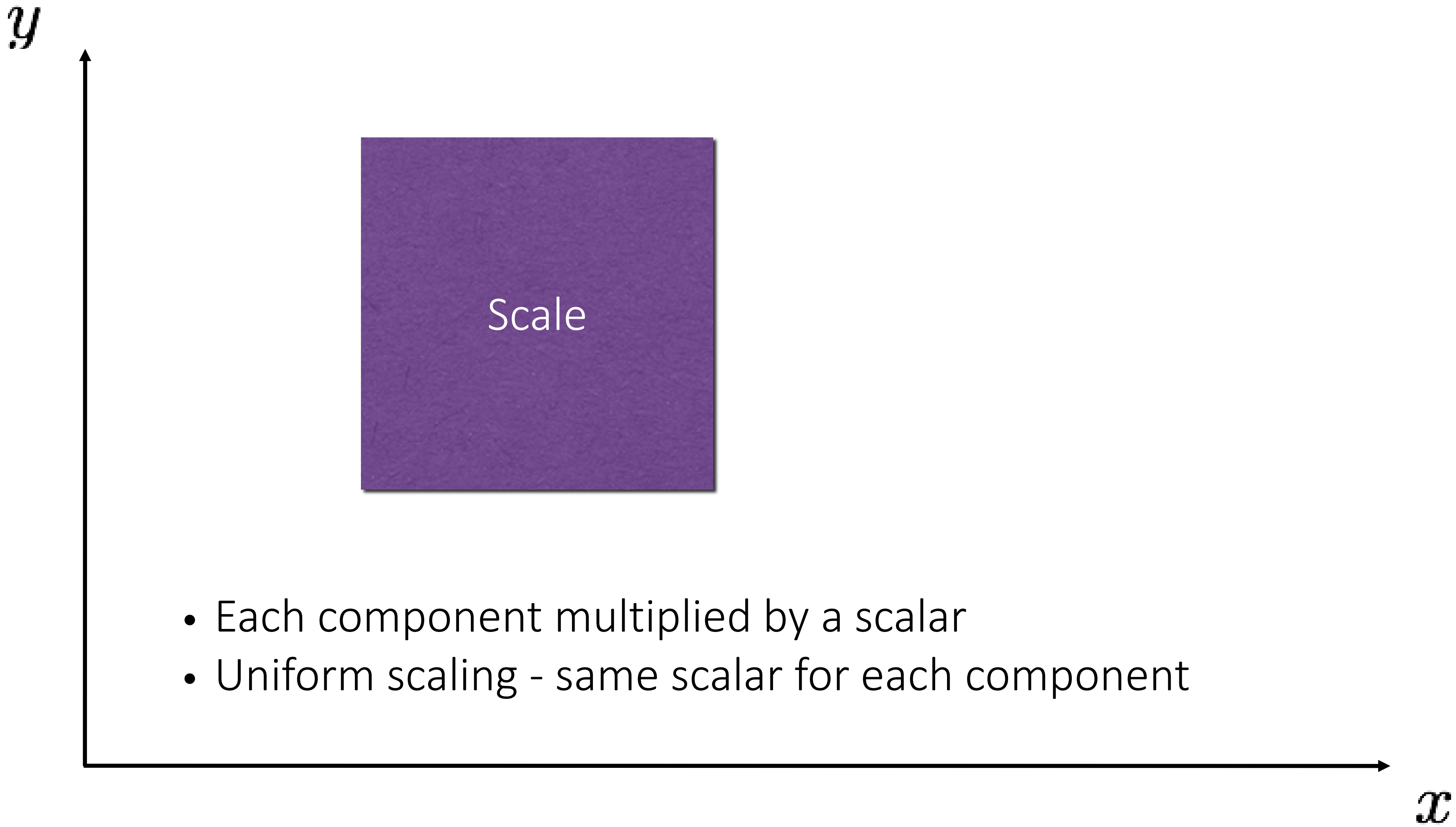
affine

perspective

cylindrical

2D planar transformations

2D planar transformations

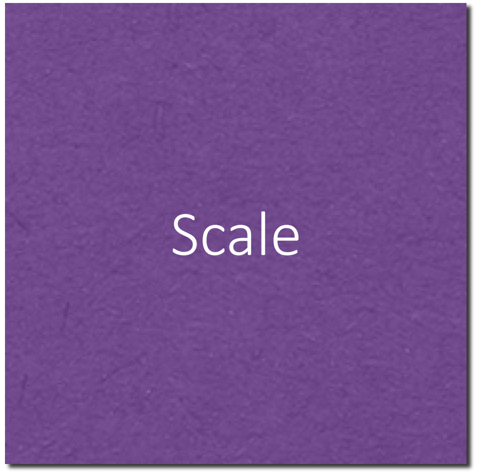


- Each component multiplied by a scalar
- Uniform scaling - same scalar for each component

2D planar transformations

y

$$x' = ax$$
$$y' = by$$

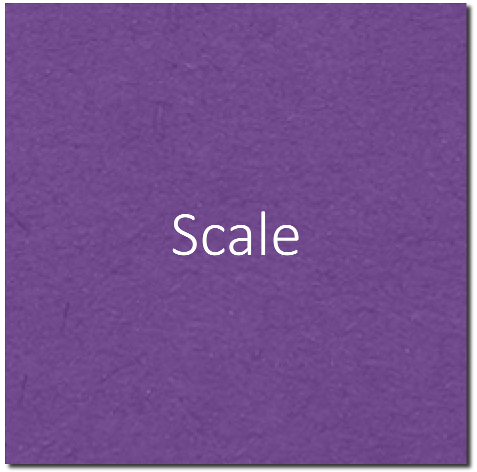


- Each component multiplied by a scalar
- Uniform scaling - same scalar for each component

x

2D planar transformations

y



$$x' = ax$$

$$y' = by$$

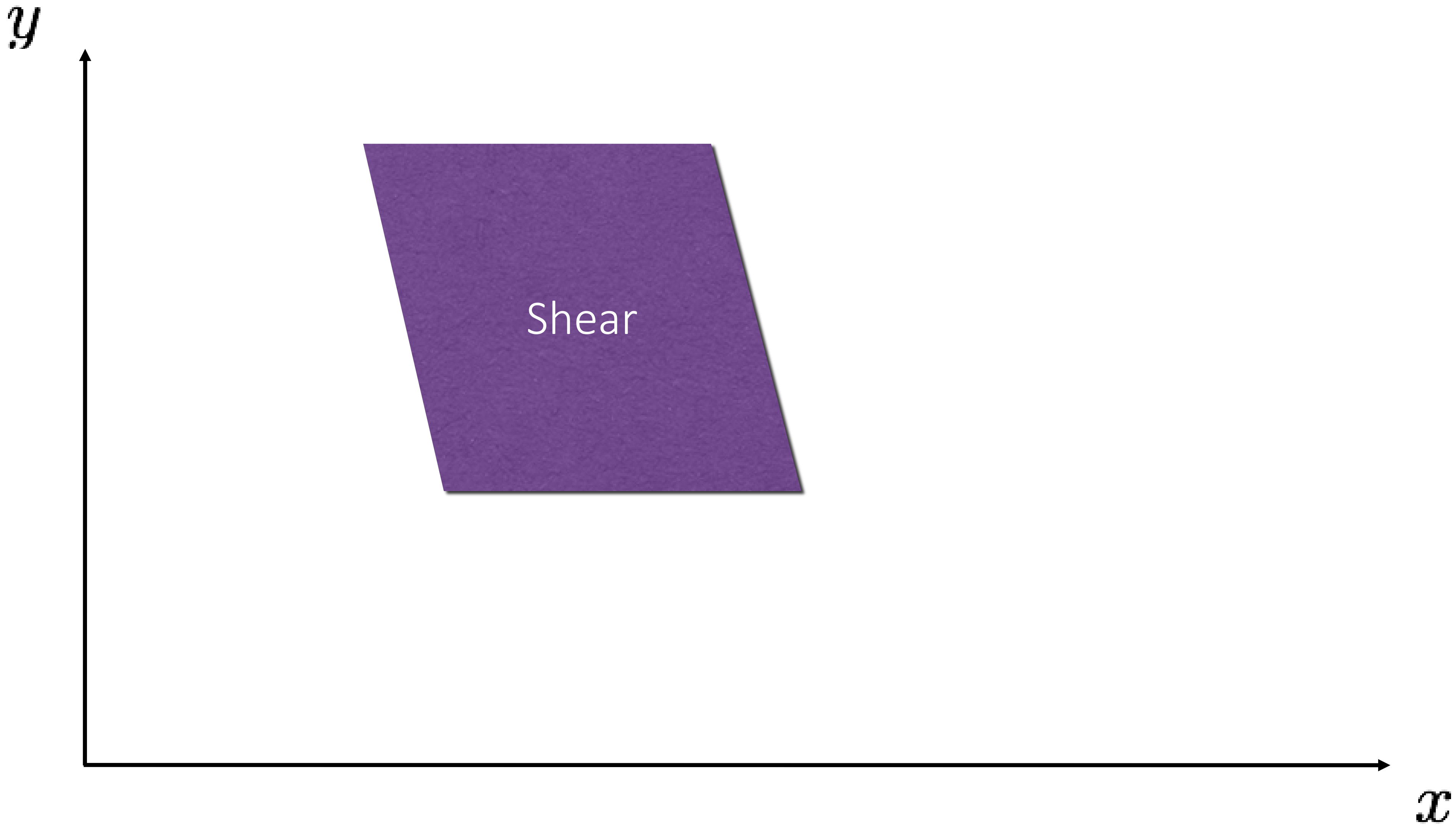
matrix representation of scaling:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}}_{\text{scaling matrix S}} \begin{bmatrix} x \\ y \end{bmatrix}$$

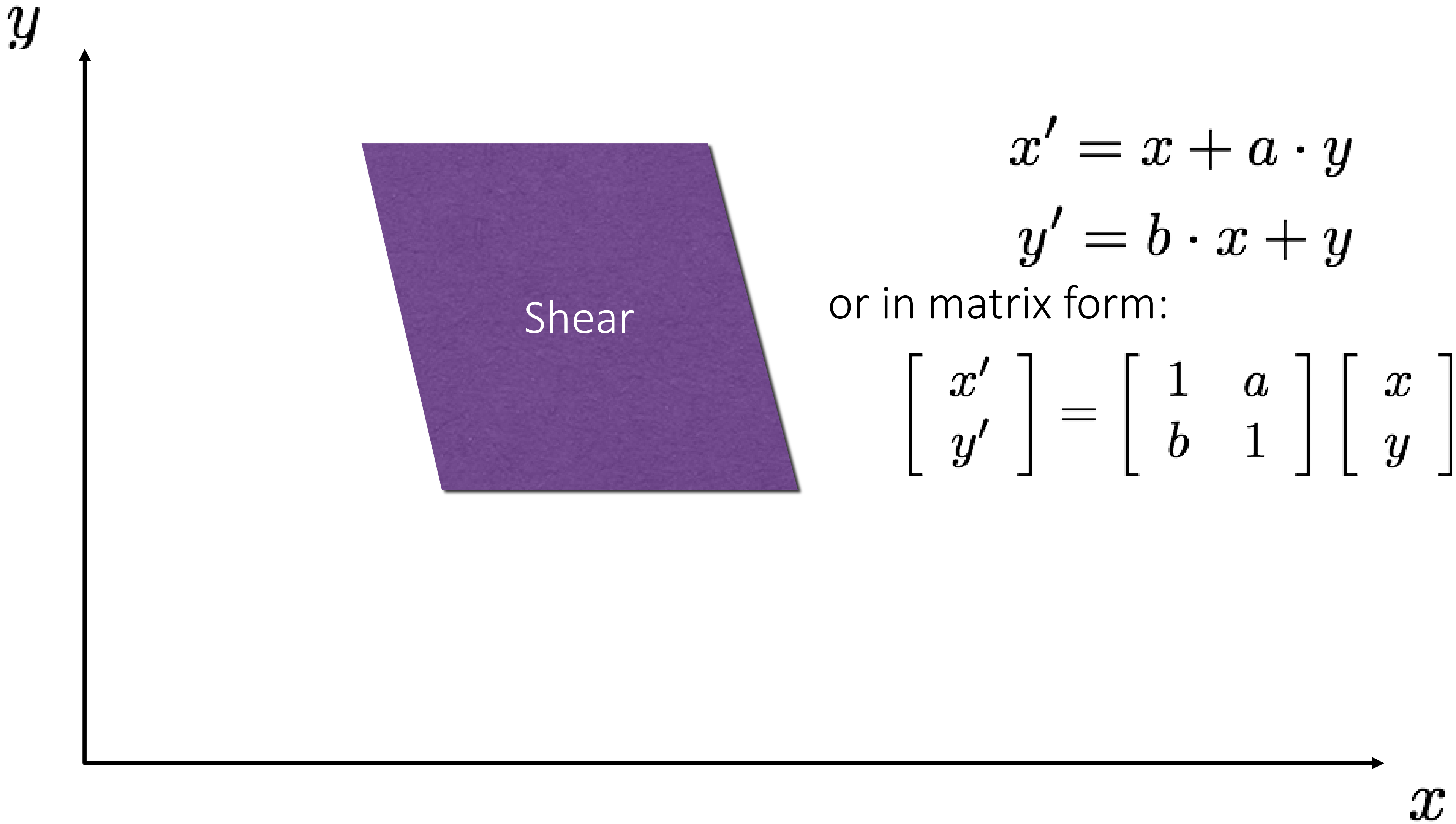
- Each component multiplied by a scalar
- Uniform scaling - same scalar for each component

x

2D planar transformations



2D planar transformations



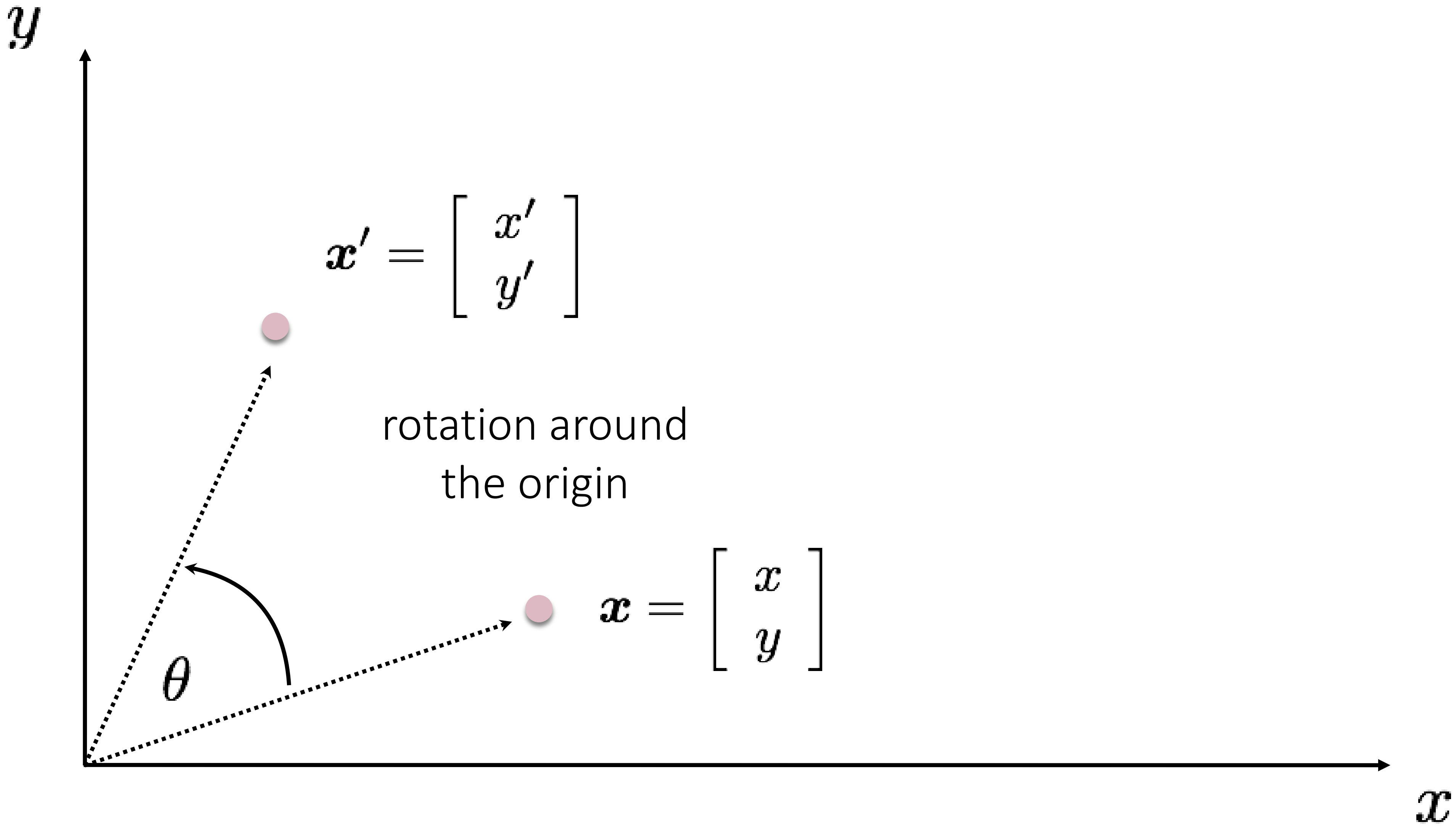
$$x' = x + a \cdot y$$

$$y' = b \cdot x + y$$

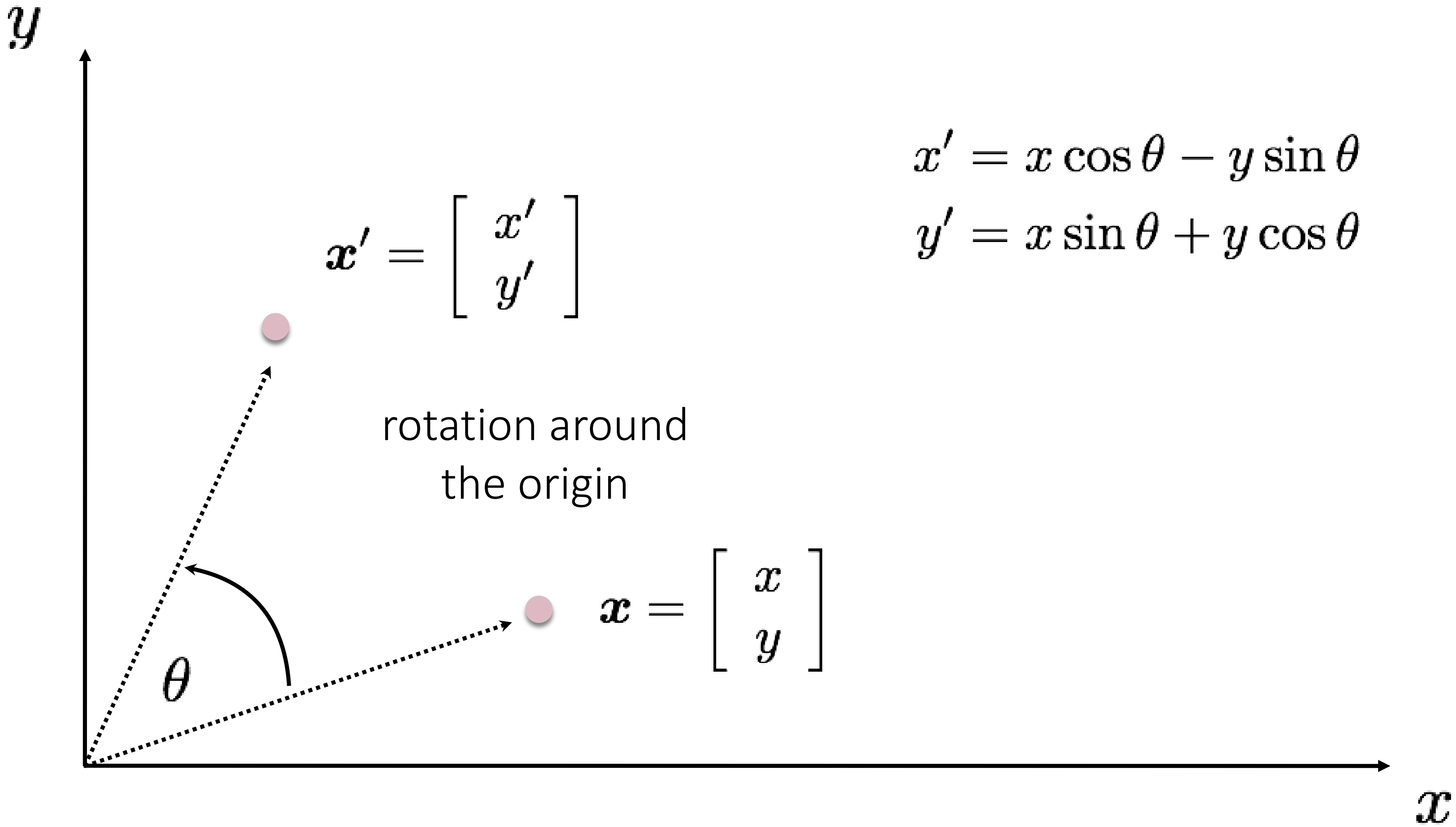
or in matrix form:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & a \\ b & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2D planar transformations

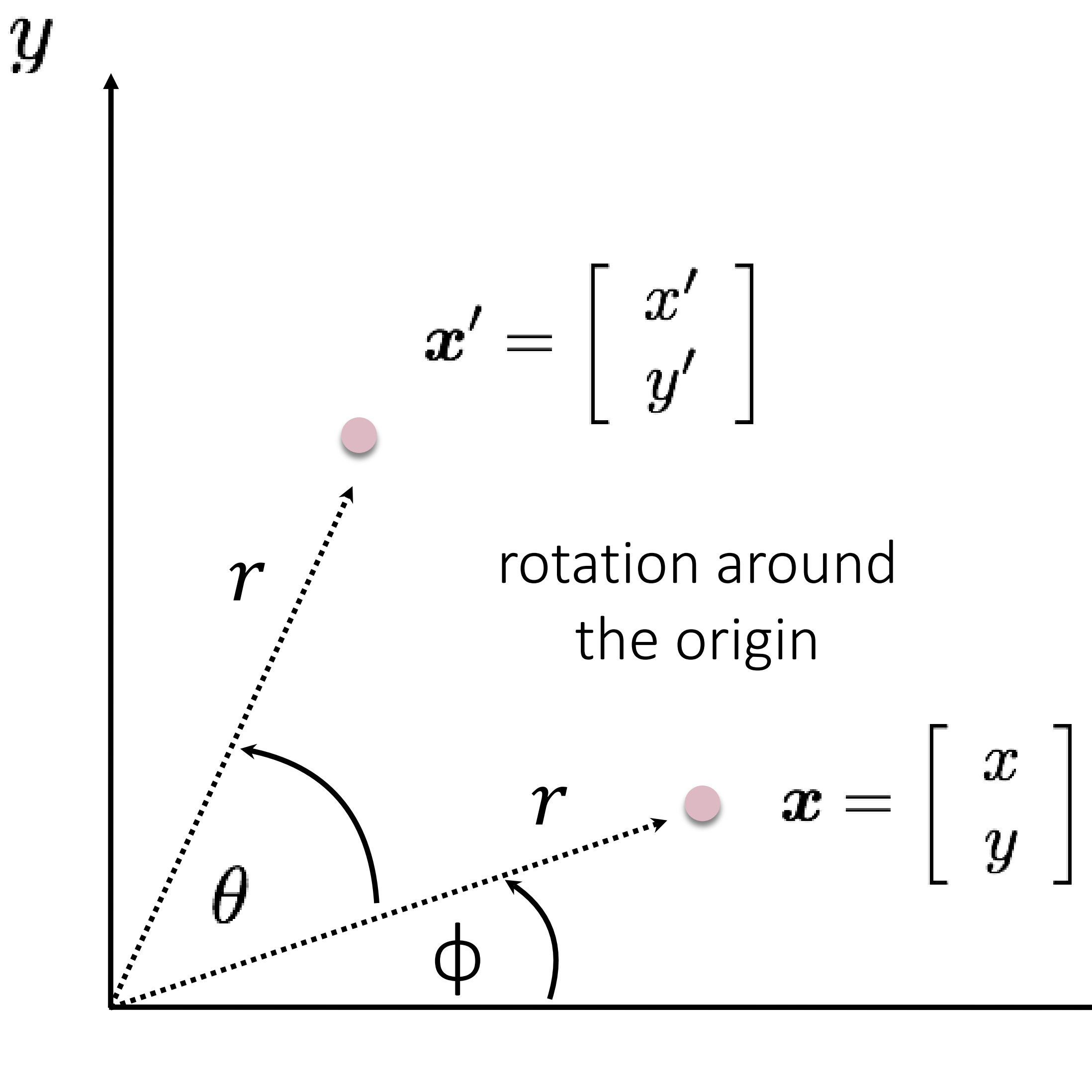


2D planar transformations



$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$

2D planar transformations



Polar coordinates...

$x = r \cos(\phi)$

$y = r \sin(\phi)$

$x' = r \cos(\phi + \theta)$

$y' = r \sin(\phi + \theta)$

Trigonometric Identity...

$x' = r \cos(\phi) \cos(\theta) - r \sin(\phi) \sin(\theta)$

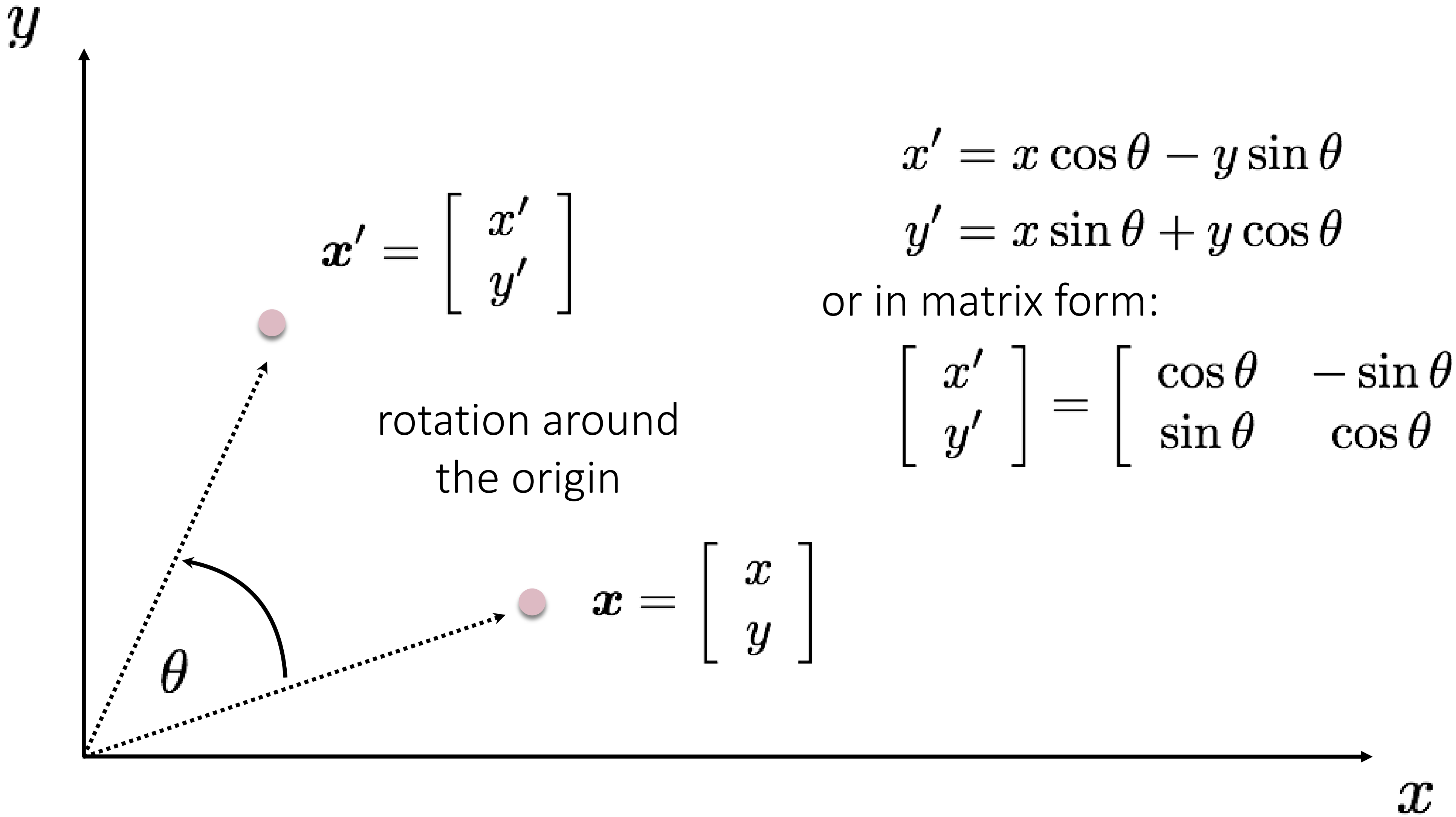
$y' = r \sin(\phi) \cos(\theta) + r \cos(\phi) \sin(\theta)$

Substitute...

$x' = x \cos(\theta) - y \sin(\theta)$

$y' = x \sin(\theta) + y \cos(\theta)$

2D planar transformations



$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$

or in matrix form:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2D planar and linear transformations

$$\boldsymbol{x}' = f(\boldsymbol{x}; p)$$

$$\downarrow$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \boldsymbol{M} \begin{bmatrix} x \\ y \end{bmatrix}$$

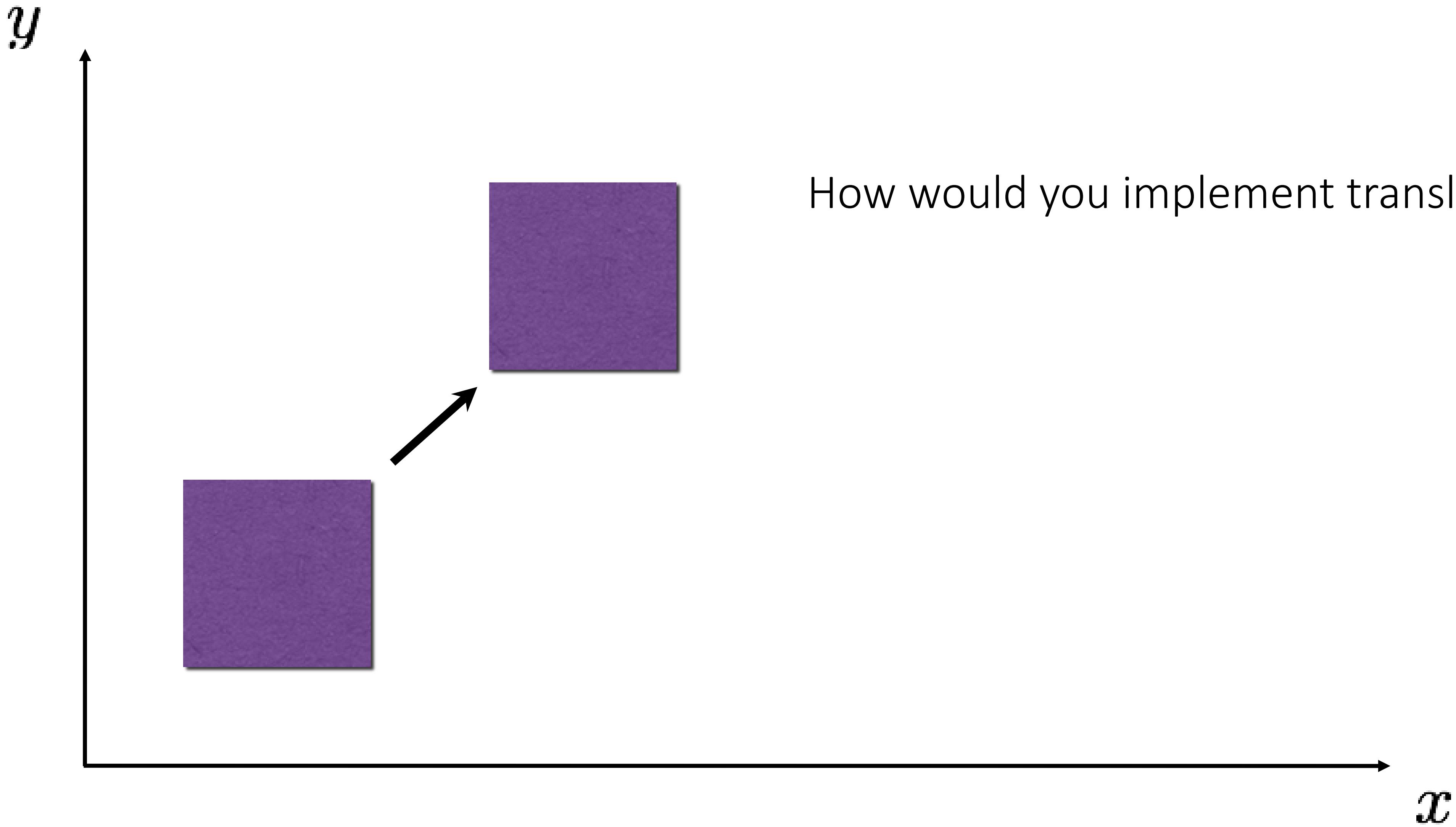
parameters $\boldsymbol{p}$

point $\boldsymbol{x}$

2D planar and linear transformations

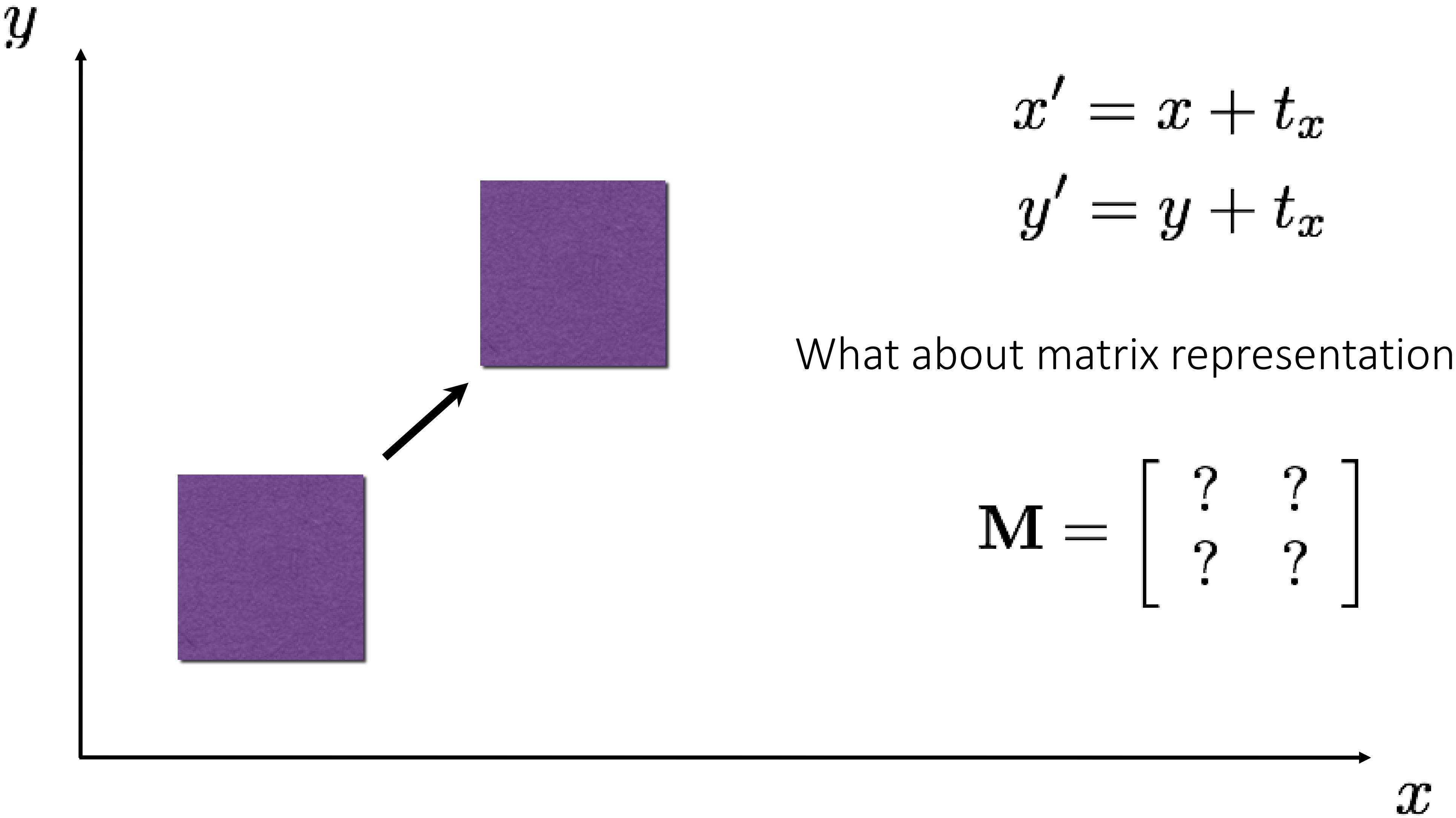
Scale $$\mathbf{M} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$$	Flip across y $$\mathbf{M} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$
Rotate $$\mathbf{M} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$	Flip across origin $$\mathbf{M} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$
Shear $$\mathbf{M} = \begin{bmatrix} 1 & s_x \\ s_y & 1 \end{bmatrix}$$	Identity $$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

2D translation



How would you implement translation?

2D translation



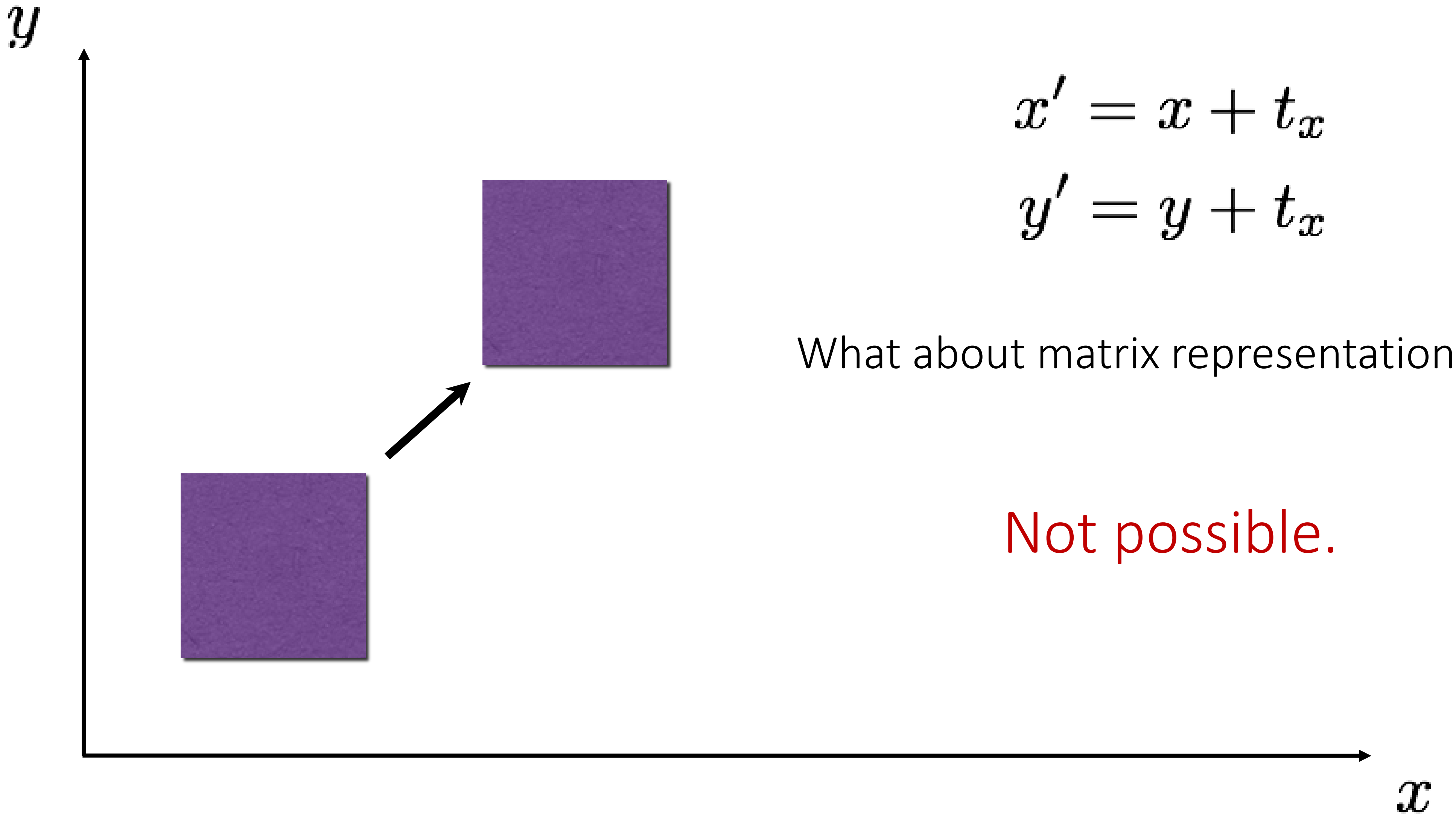
$$x' = x + t_x$$

$$y' = y + t_x$$

What about matrix representation?

$$\mathbf{M} = \left[\begin{array}{cc} ? & ? \\ ? & ? \end{array} \right]$$

2D translation



$$x' = x + t_x$$
$$y' = y + t_x$$

What about matrix representation?

Not possible.

Projective geometry 101

Homogeneous coordinates

heterogeneous coordinates

homogeneous coordinates

$$\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

add a 1 here

- Represent 2D point with a 3D vector

Homogeneous coordinates

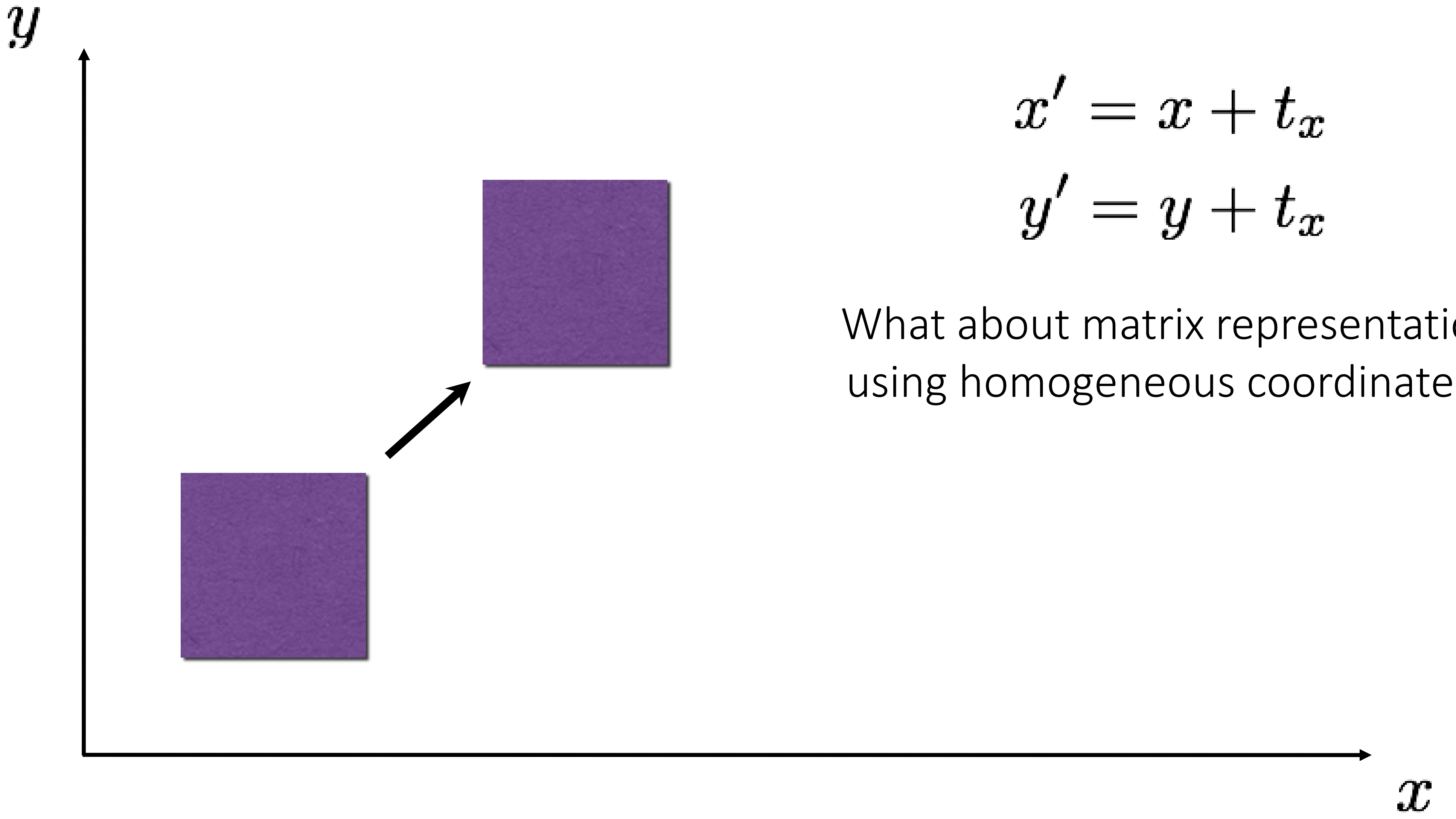
heterogeneous coordinates

homogeneous coordinates

$$\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} ax \\ ay \\ a \end{bmatrix}$$

- Represent 2D point with a 3D vector
- 3D vectors are only defined up to scale

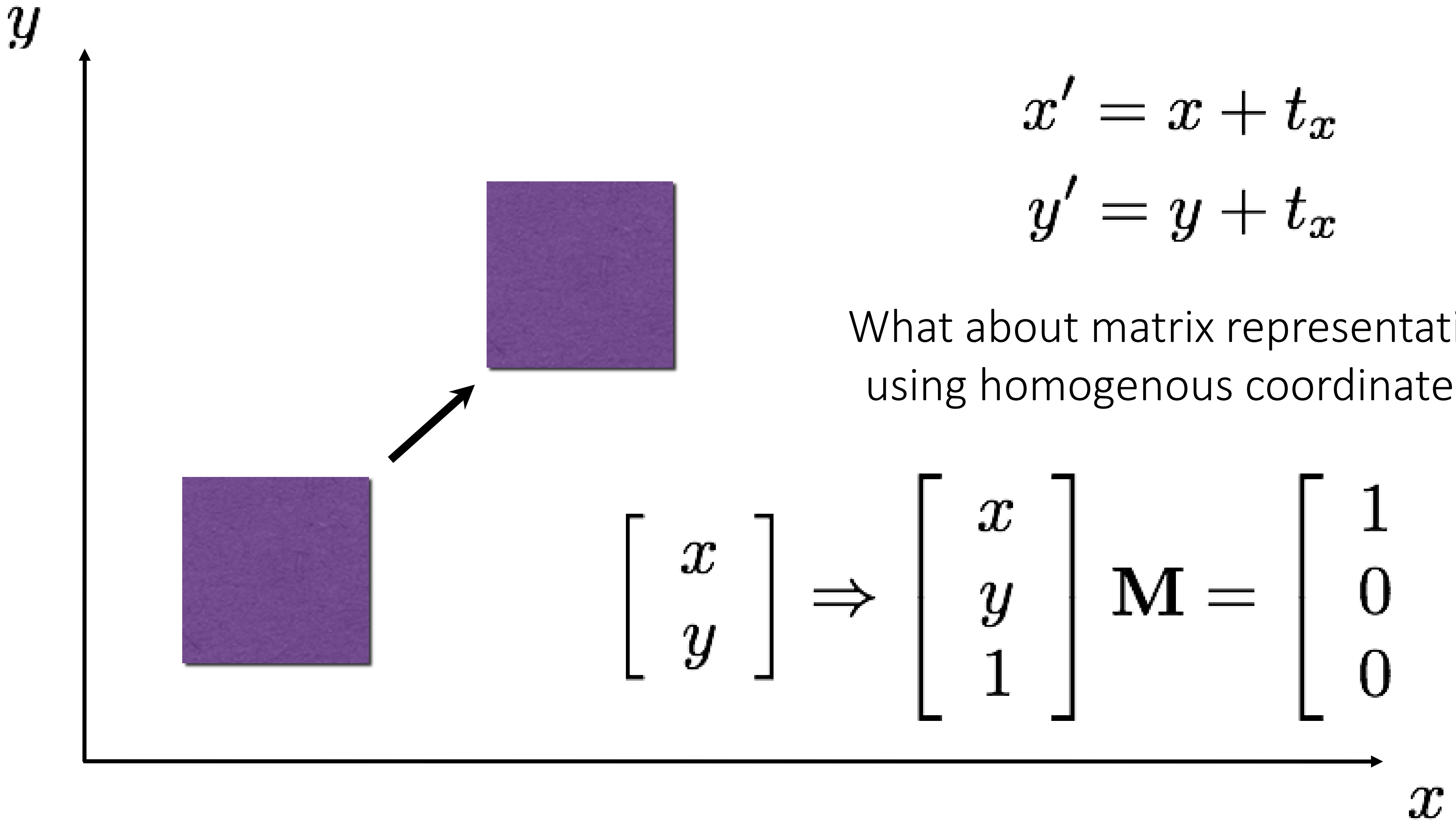
2D translation



$$x' = x + t_x$$
$$y' = y + t_x$$

What about matrix representation using homogeneous coordinates?

2D translation



$$x' = x + t_x$$

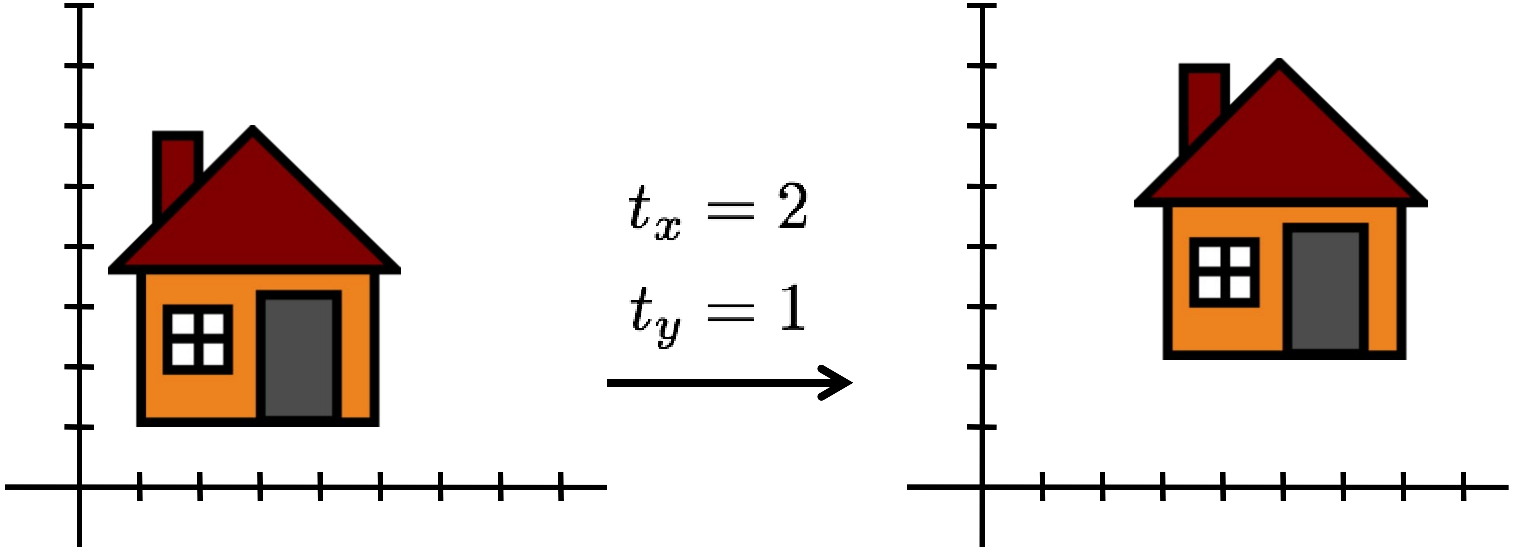
$$y' = y + t_x$$

What about matrix representation
using homogenous coordinates?

$$\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \quad \mathbf{M} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

2D translation using homogeneous coordinates

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix}$$



Homogeneous coordinates

Conversion:

- heterogeneous → homogeneous

$$\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- homogeneous → heterogeneous

$$\begin{bmatrix} x \\ y \\ w \end{bmatrix} \Rightarrow \begin{bmatrix} x/w \\ y/w \end{bmatrix}$$

- scale invariance

$$\begin{bmatrix} x & y & w \end{bmatrix}^\top = \lambda \begin{bmatrix} x & y & w \end{bmatrix}^\top$$

Special points:

- point at infinity

$$\begin{bmatrix} x & y & 0 \end{bmatrix}$$

- undefined

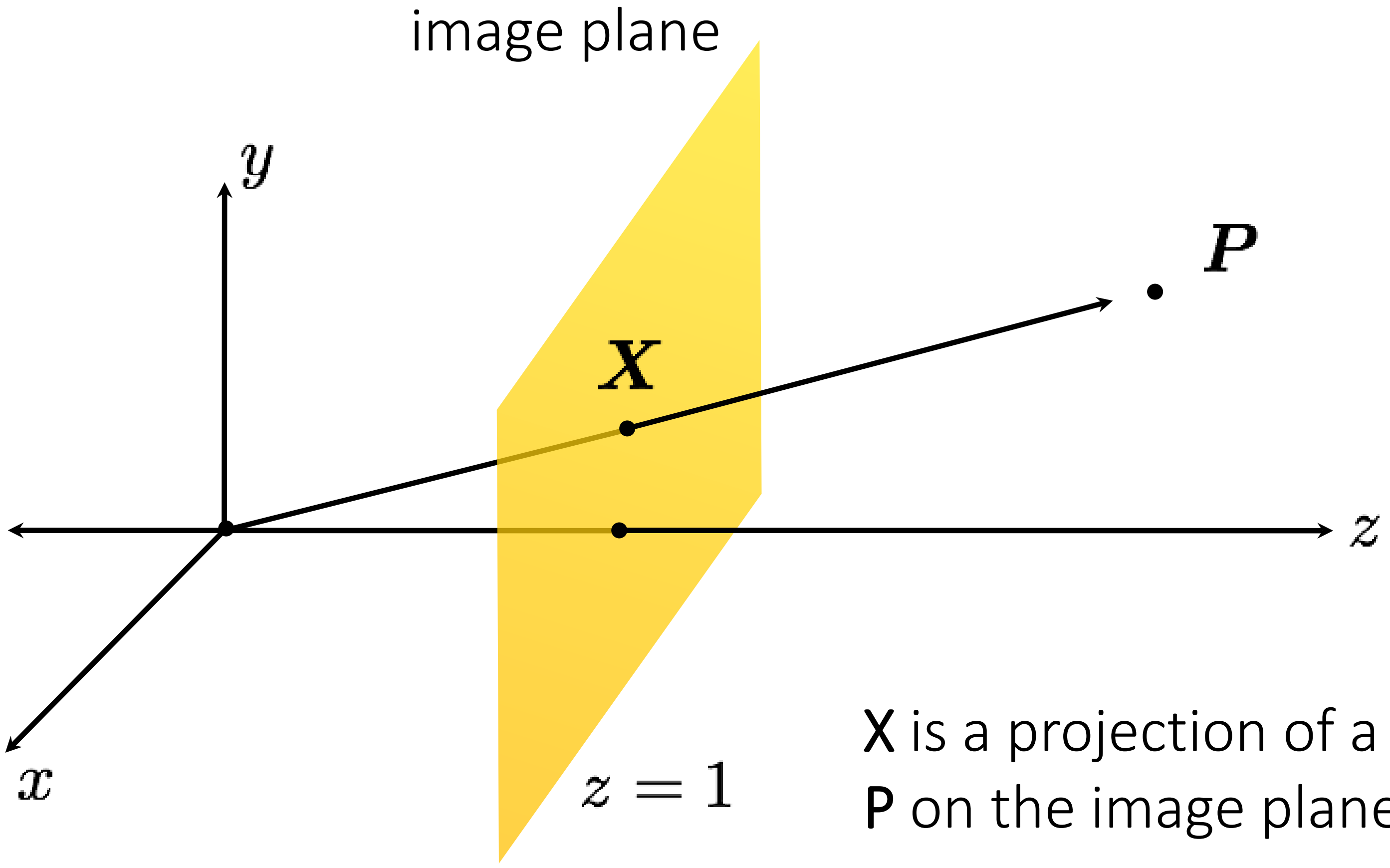
$$\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

Projective geometry

image point in pixel coordinates $\boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix}$

$$\Updownarrow$$

image point in homogeneous coordinates $\boldsymbol{X} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$



X is a projection of a point **P** on the image plane

What does scaling **X** correspond to?

Transformations in projective geometry

2D transformations in heterogeneous coordinates

Re-write these transformations as 3x3 matrices:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

translation

$$\begin{bmatrix} \boldsymbol{x}' \\ \boldsymbol{y}' \\ 1 \end{bmatrix} = \begin{bmatrix} & & \\ & ? & \\ & & \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ 1 \end{bmatrix}$$

scaling

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} & & \\ & ? & \\ & & \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

rotation

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} & & \\ & ? & \\ & & \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

shearing

2D transformations in heterogeneous coordinates

Re-write these transformations as 3x3 matrices:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

translation

$$\begin{bmatrix} \boldsymbol{x}' \\ \boldsymbol{y}' \\ 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{s}_{\boldsymbol{x}} & 0 & 0 \\ 0 & \boldsymbol{s}_{\boldsymbol{y}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ 1 \end{bmatrix}$$

scaling

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} & & \\ & ? & \\ & & \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

rotation

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} & & \\ & ? & \\ & & \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

shearing

2D transformations in heterogeneous coordinates

Re-write these transformations as 3x3 matrices:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

translation

$$\begin{bmatrix} \boldsymbol{x}' \\ \boldsymbol{y}' \\ 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{s_x} & 0 & 0 \\ 0 & \boldsymbol{s_y} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ 1 \end{bmatrix}$$

scaling

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} & & \\ & ? & \\ & & \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

rotation

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & \beta_x & 0 \\ \beta_y & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

shearing

2D transformations in heterogeneous coordinates

Re-write these transformations as 3x3 matrices:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

translation

$$\begin{bmatrix} \boldsymbol{x}' \\ \boldsymbol{y}' \\ 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{s}_{\boldsymbol{x}} & 0 & 0 \\ 0 & \boldsymbol{s}_{\boldsymbol{y}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ 1 \end{bmatrix}$$

scaling

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\Theta & -\sin\Theta & 0 \\ \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

rotation

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & \beta_x & 0 \\ \beta_y & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

shearing

Matrix composition

Transformations can be combined by matrix multiplication:

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \left(\begin{bmatrix} 1 & 0 & tx \\ 0 & 1 & ty \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\Theta & -\sin\Theta & 0 \\ \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} sx & 0 & 0 \\ 0 & sy & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

p' = ? ? ? **p**

Matrix composition

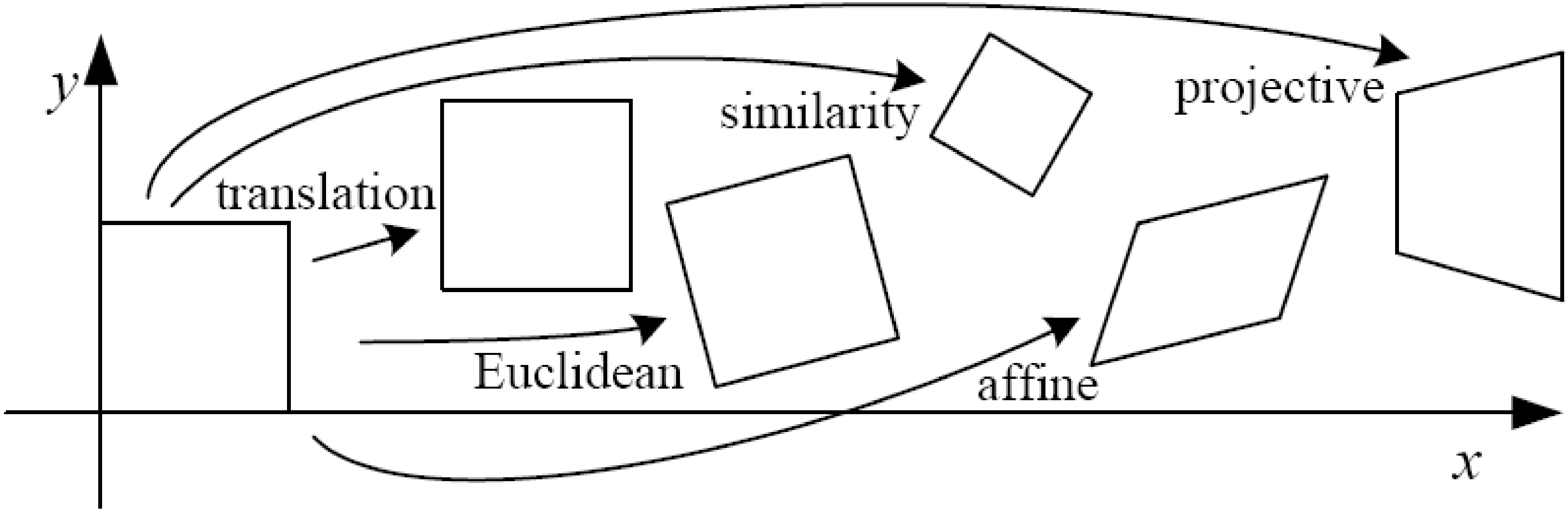
Transformations can be combined by matrix multiplication:

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \left(\begin{bmatrix} 1 & 0 & tx \\ 0 & 1 & ty \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\Theta & -\sin\Theta & 0 \\ \sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} sx & 0 & 0 \\ 0 & sy & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

p' = translation(t_x,t_y) rotation(θ) scale(s,s) **p**

Classification of 2D transformations

Classification of 2D transformations

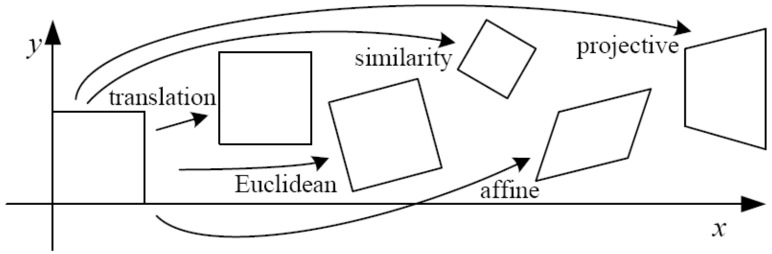


Classification of 2D transformations

Name	Matrix	# D.O.F.
translation	$\begin{bmatrix} \boldsymbol{I} \mid \boldsymbol{t} \end{bmatrix}$	?
rigid (Euclidean)	$\begin{bmatrix} \boldsymbol{R} \mid \boldsymbol{t} \end{bmatrix}$	?
similarity	$\begin{bmatrix} s\boldsymbol{R} \mid \boldsymbol{t} \end{bmatrix}$	?
affine	$\begin{bmatrix} \boldsymbol{A} \end{bmatrix}$	?
projective	$\begin{bmatrix} \tilde{\boldsymbol{H}} \end{bmatrix}$	?

Classification of 2D transformations

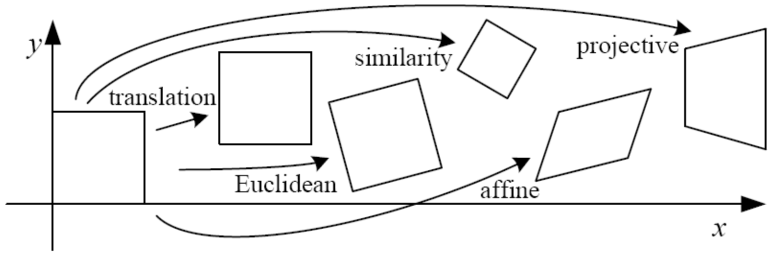
Translation: $$\begin{bmatrix} 1 & 0 & t_1 \\ 0 & 1 & t_2 \\ 0 & 0 & 1 \end{bmatrix}$$



Classification of 2D transformations

Euclidean (rigid):
rotation + translation

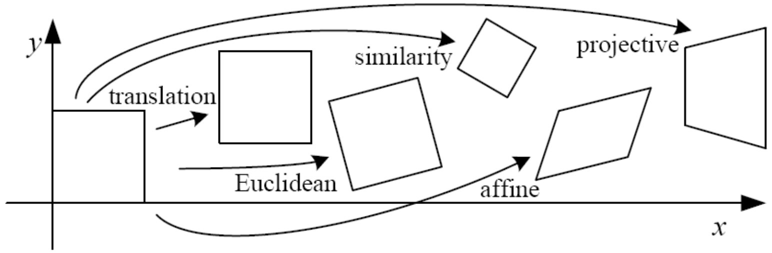
$$\begin{bmatrix} r_1 & r_2 & r_3 \\ r_4 & r_5 & r_6 \\ 0 & 0 & 1 \end{bmatrix}$$



Classification of 2D transformations

Euclidean (rigid):
rotation + translation

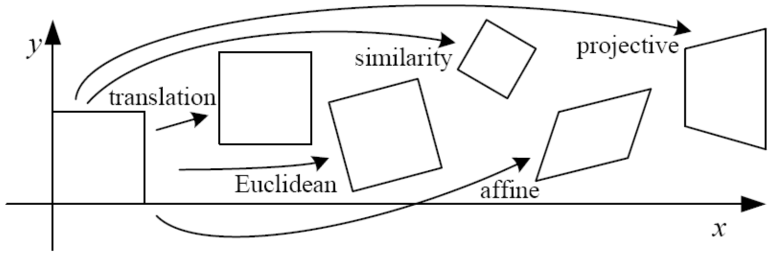
$$\begin{bmatrix} \cos\theta & -\sin\theta & r_3 \\ \sin\theta & \cos\theta & r_6 \\ 0 & 0 & 1 \end{bmatrix}$$



Classification of 2D transformations

Similarity:
uniform scaling + rotation
+ translation

$$\begin{bmatrix} r_1 & r_2 & r_3 \\ r_4 & r_5 & r_6 \\ 0 & 0 & 1 \end{bmatrix}$$

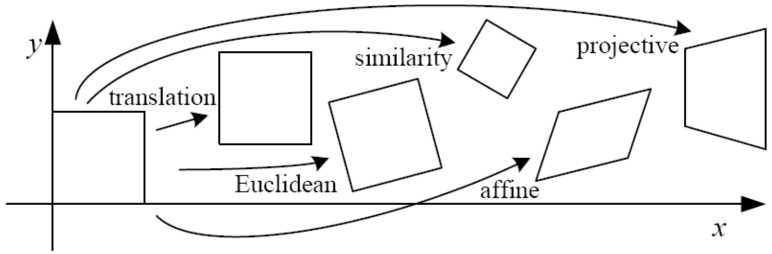


Classification of 2D transformations

multiply these four by scale **s**

Similarity:
uniform scaling + rotation
+ translation

$$\begin{bmatrix} \boxed{\begin{matrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{matrix}} & \begin{matrix} r_3 \\ r_6 \end{matrix} \\ \begin{matrix} 0 & \quad 0 \end{matrix} & 1 \end{bmatrix}$$



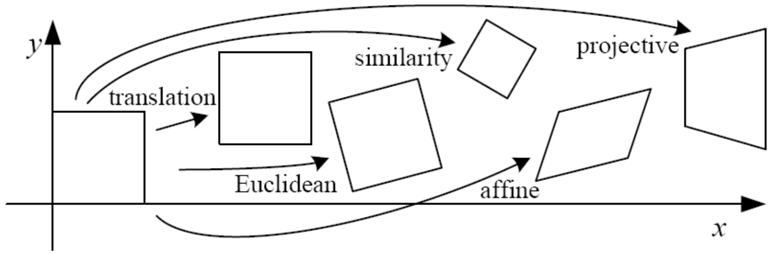
Classification of 2D transformations

Affine transform

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos(-\Phi) & -\sin(-\Phi) \\ \sin(-\Phi) & \cos(-\Phi) \end{bmatrix} \cdots$$
$$\cdots \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} \cos\Phi & -\sin\Phi \\ \sin\Phi & \cos\Phi \end{bmatrix}$$

Linear part can be decomposed



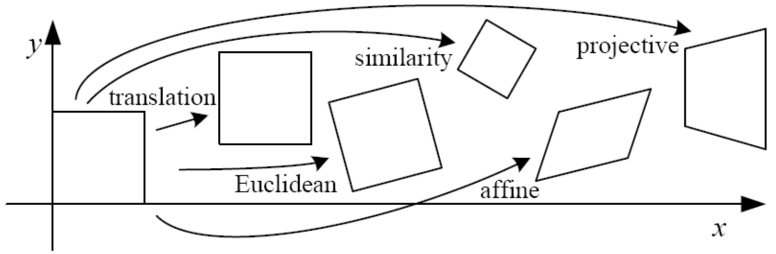
Classification of 2D transformations

Affine transform $$\mathbf{x}' = H_A \mathbf{x} = \begin{bmatrix} A & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix} \mathbf{x}$$

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos(-\Phi) & -\sin(-\Phi) \\ \sin(-\Phi) & \cos(-\Phi) \end{bmatrix} \cdots$$
$$\cdots \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} \cos\Phi & -\sin\Phi \\ \sin\Phi & \cos\Phi \end{bmatrix}$$

Linear part can be decomposed

$$A = R(\theta)R(-\Phi)D(\lambda_1, \lambda_2)R(\Phi)$$



Affine transformations

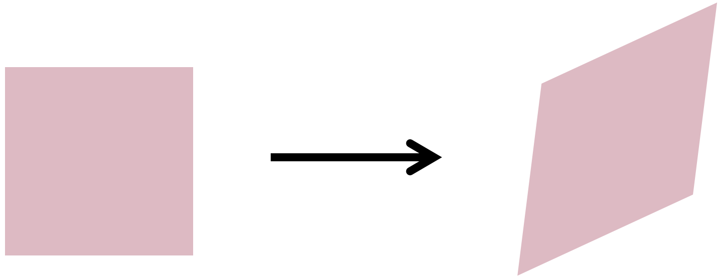
Affine transformations are combinations of

- arbitrary (4-DOF) linear transformations; and
- translations

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

Properties of affine transformations:

- origin does not necessarily map to origin
- lines map to lines
- parallel lines map to parallel lines
- ratios are preserved
- compositions of affine transforms are also affine transforms

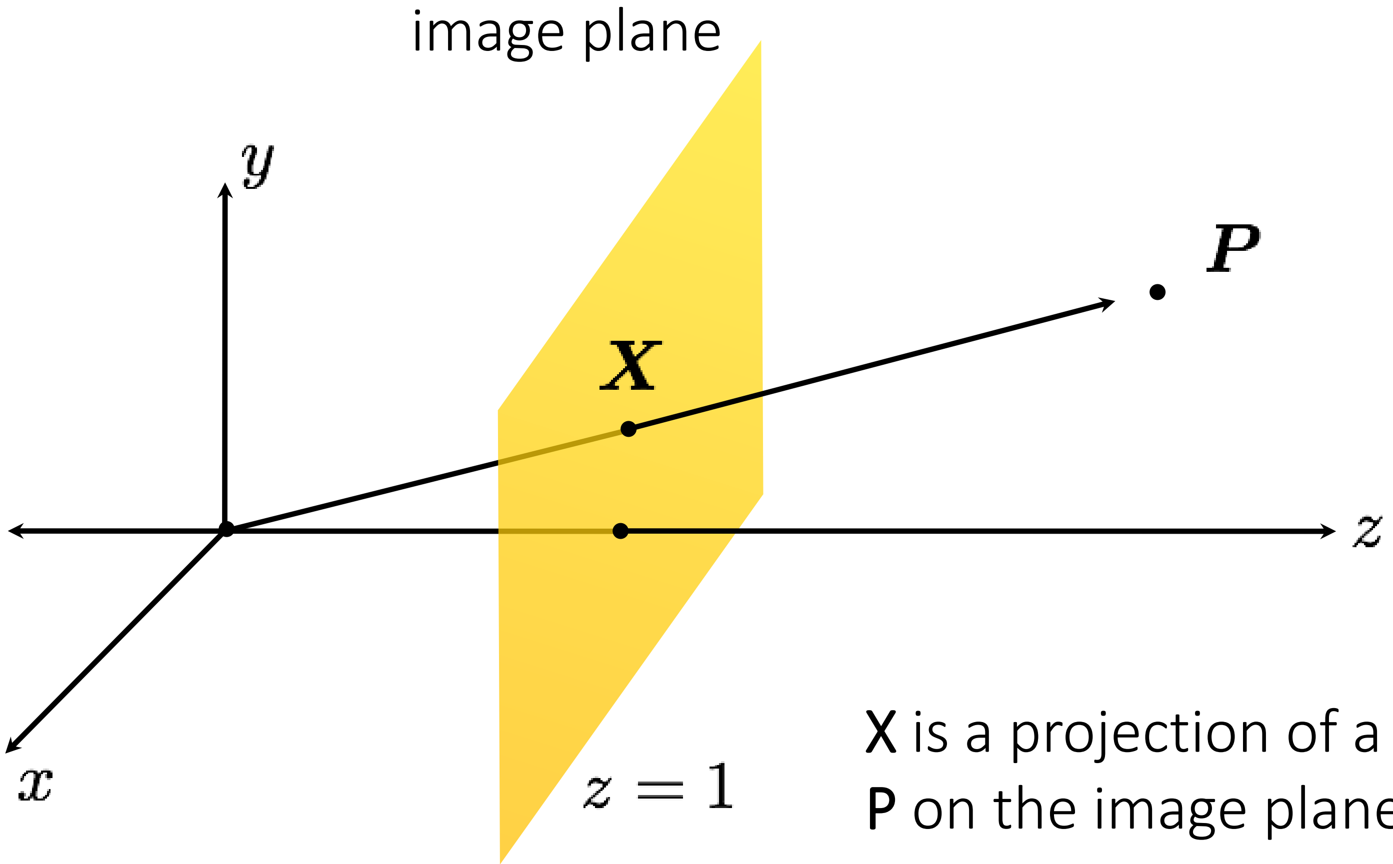


Projective transformations

image point in pixel coordinates $\boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix}$

$\Updownarrow$

image point in heterogeneous coordinates $\boldsymbol{X} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$



X is a projection of a point P on the image plane

Projective transformations

Projective transformations are combinations of

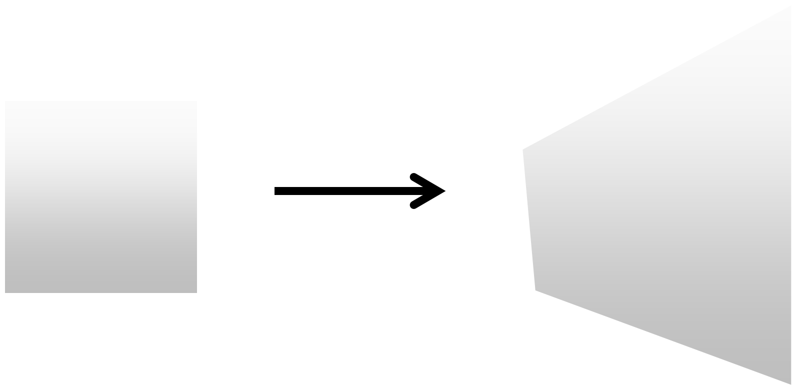
- affine transformations; and
- projective wraps

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

How many degrees of freedom?

Properties of projective transformations:

- origin does not necessarily map to origin
- lines map to lines
- parallel lines do not necessarily map to parallel lines
- ratios are not necessarily preserved
- compositions of projective transforms are also projective transforms



Projective transforms = 8Dof

$$k_{p2}\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} k_{p1}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \frac{k_{p1}}{k_{p2}}\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \frac{k_{p1}}{k_{p2}}\begin{bmatrix} a_{11}/a_{33} & a_{12}/a_{33} & a_{13}/a_{33} \\ a_{21}/a_{33} & a_{22}/a_{33} & a_{23}/a_{33} \\ a_{31}/a_{33} & a_{32}/a_{33} & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = k\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Projective transformations

Projective transformations are combinations of

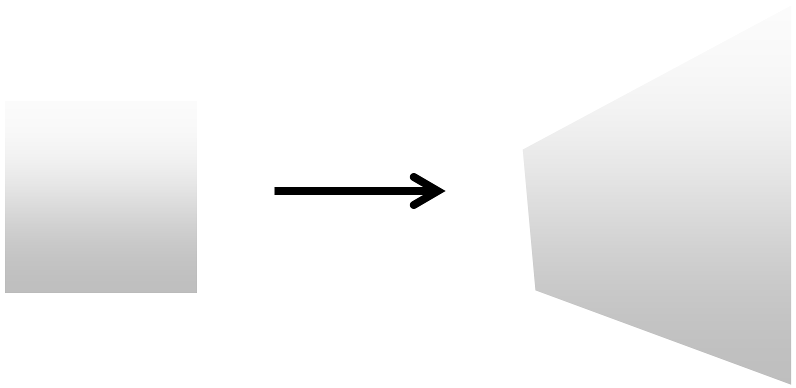
- affine transformations; and
- projective wraps

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

8 DOF: vectors (and therefore matrices) are defined up to scale)

Properties of projective transformations:

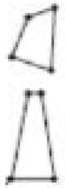
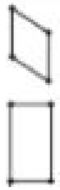
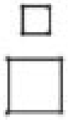
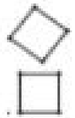
- origin does not necessarily map to origin
- lines map to lines
- parallel lines do not necessarily map to parallel lines
- ratios are not necessarily preserved
- compositions of projective transforms are also projective transforms



Classification of 2D transformations

Name	Matrix	# D.O.F.
translation	$\left[\begin{array}{c\|c} \boldsymbol{I} & \boldsymbol{t} \end{array}\right]$	2
rigid (Euclidean)	$\left[\begin{array}{c\|c} \boldsymbol{R} & \boldsymbol{t} \end{array}\right]$	3
similarity	$\left[\begin{array}{c\|c} s\boldsymbol{R} & \boldsymbol{t} \end{array}\right]$	3
affine	$\left[\begin{array}{c} \boldsymbol{A} \end{array}\right]$	6
projective	$\left[\begin{array}{c} \tilde{\boldsymbol{H}} \end{array}\right]$	8

Properties

Group	Matrix	Distortion	Invariant properties
Projective 8 dof	$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$		Concurrency, collinearity, **order of contact**: intersection (1 pt contact); tangency (2 pt contact); inflections (3 pt contact with line); tangent discontinuities and cusps. cross ratio (ratio of ratio of lengths).
Affine 6 dof	$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Parallelism, ratio of areas, ratio of lengths on collinear or parallel lines (e.g. midpoints), linear combinations of vectors (e.g. centroids). The line at infinity, $\mathbf{l}_\infty$.
Similarity 4 dof	$\begin{bmatrix} sr_{11} & sr_{12} & t_x \\ sr_{21} & sr_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Ratio of lengths, angle. The circular points, **I**, **J** (see section 2.7.3).
Euclidean 3 dof	$\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Length, area