First-Order Logic: Review

RDFS/OWL Smantics

- The semantics of RDFS and OWL are based on First Order Logic
- Advantages:
  - Familiar, well defined, well understood, expressive, powerful
  - Good procedures/tools for inference
- Disadvantages
  - No agreement on how to extend for probabilities, fuzzy representations, higher order logics, etc.
  - Hard to process in parallel

First-order logic

- First-order logic (FOL) models the world in terms of
  - Objects, which are things with individual identities
  - Properties of objects that distinguish them from others
  - Relations that hold among sets of objects
  - Functions, which are a subset of relations where there is only one "value" for any given "input"
- Examples:
  - Objects: Students, lectures, companies, cars ...
  - Relations: Brother-of, bigger-than, outside, part-of, has-color, occurs-after, owns, visits, precedes, ...
  - Properties: blue, oval, even, large, ...
  - Functions: father-of, best-friend, second-half, more-than ...

User provides

- Constant symbols representing individuals in the world
  - Mary, 3, green
- Function symbols, map individuals to individuals
  - father_of(Mary) = John
  - color_of(Sky) = Blue
- Predicate symbols, map individuals to **truth values**
  - greater(5,3)
  - green(Grass)
  - color(Grass, Green)

FOL Provides

- Truth values
  - True, False
- Variable symbols
  - E.g., x, y, foo
- Connectives
  - Same as in propositional logic: not ($\neg$), and ($\wedge$), or ($\vee$), implies ($\rightarrow$), iff ($\leftrightarrow$)
- Quantifiers
  - Universal $\forall$x or (Ax)
  - Existential $\exists$x or (Ex)

Sentences: built from terms and atoms

- A term (denoting a real-world individual) is a constant symbol, variable symbol, or n-place function of n terms, e.g.:
  - Constants: john, umbc
  - Variables: x, y, z
  - Functions: mother_of(john), phone(mother(x))
- Ground terms have no variables in them
  - Ground: john, father_of(father_of(john))
  - Not Ground: father_of(X)

Sentences: built from terms and atoms

- An atomic sentence (which has value true or false) is an n-place predicate of n terms, e.g.:
  - green(Kermit))
  - between(Philadelphia, Baltimore, DC)
  - loves(X, mother(X))
- A complex sentence is formed from atomic sentences connected by logical connectives:

  $\neg P$, $P \vee Q$, $P \wedge Q$, $P \rightarrow Q$, $P \leftrightarrow Q$

  where P and Q are sentences

Sentences: built from terms and atoms

- quantified sentences adds quantifiers $\forall$ and $\exists$
  - $\forall$x loves(x, mother(x))
  - $\exists$x number(x) $\wedge$ greater(x, 100), prime(x)
- A well-formed formula (wff) is a sentence containing no "free" variables, i.e., all variables are "bound" by either a universal or existential quantifiers

  ($\forall$x)P(x,y) has x bound as a universally quantified variable, but y is free

A BNF for FOL

```
S := <Sentence> ;
<Sentence> := <AtomicSentence> |
          <Sentence> <Connective> <Sentence> |
          <Quantifier> <Variable>,... <Sentence> |
          "NOT" <Sentence> |
          "(" <Sentence> ")";
<AtomicSentence> := <Predicate> "(" <Term>, ... ")" |
                    <Term> "=" <Term>;
<Term> := <Function> "(" <Term>, ... ")" |
          <Constant> |
          <Variable>;
<Connective> := "AND" | "OR" | "IMPLIES" | "EQUIVALENT";
<Quantifier> := "EXISTS" | "FORALL" ;
<Constant> := "A" | "X1" | "John" | ... ;
<Variable> := "a" | "x" | "s" | ... ;
<Predicate> := "Before" | "HasColor" | "Raining" | ... ;
<Function> := "Mother" | "LeftLegOf" | ... ;
```

Quantifiers

- Universal quantification
  - $(\forall x)P(x)$ means P holds for all values of x in domain associated with variable
  - E.g., $(\forall x)$ dolphin(x) $\rightarrow$ mammal(x)
- Existential quantification
  - $(\exists x)P(x)$ means P holds for some value of x in domain associated with variable
  - E.g., $(\exists x)$ mammal(x) $\wedge$ lays_eggs(x)
  - This lets us make a statement about some object without naming it

Quantifiers (1)

- Universal quantifiers often used with implies to form rules:

  $(\forall x)$ student(x) $\rightarrow$ smart(x) means "All students are smart"

- Universal quantification rarely used to make blanket statements about every individual in the world:

  $(\forall x)$ student(x) $\wedge$ smart(x) means "Everyone in the world is a student and is smart"

Quantifiers (2)

- Existential quantifiers usually used with "and" to specify
  a list of properties about an individual:
  - $(\exists x)$ student(x) $\wedge$ smart(x) means "There is a student who is smart"
- Common mistake: represent this in FOL as:
  - $(\exists x)$ student(x) $\rightarrow$ smart(x)
- What does this sentence mean?
  - ??

Quantifier Scope

- FOL sentences have structure, like programs
- In particular, the variables in a sentence have a scope
- For example, suppose we want to say
  - "everyone who is alive loves someone"
  - $(\forall x)\ \text{alive}(x) \rightarrow (\exists y)\ \text{loves}(x,y)$
- Here's how we scope the variables

$$(\forall x)\ \text{alive}(x) \rightarrow (\exists y)\ \text{loves}(x,y)$$

Scope of x

Scope of y

Quantifier Scope

- Switching order of universal quantifiers does not change the meaning
  - $(\forall x)(\forall y)P(x,y) \leftrightarrow (\forall y)(\forall x)\ P(x,y)$
  - "Dogs hate cats" (i.e., "all dogs hate all cats")
- You can switch order of existential quantifiers
  - $(\exists x)(\exists y)P(x,y) \leftrightarrow (\exists y)(\exists x)\ P(x,y)$
  - "A cat killed a dog"
- Switching order of universal and existential quantifiers does change meaning:
  - Everyone likes someone: $(\forall x)(\exists y)$ likes(x,y)
  - Someone is liked by everyone: $(\exists y)(\forall x)$ likes(x,y)

Procedural example 1

```
def verify1():
    # Everyone likes someone: (∀x)(∃y) likes(x,y)
    for x in people():
        found = False
        for y in people():
            if likes(x,y):
                found = True
                break
        if not Found:
            return False
    return True
```

Every person has at least one individual that they like.

Procedural example 2

```
def verify2():
    # Someone is liked by everyone: (∃y)(∀x) likes(x,y)
    for y in people():
        found = True
        for x in people():
            if not likes(x,y):
                found = False
                break
        if found
            return True
    return False
```

There is a person who is liked by every person in the universe.

Connections between ∀ and ∃

- We can relate sentences involving ∀ and ∃ using extensions to De Morgan's laws:
  1. $(\forall x)\ \neg P(x) \leftrightarrow \neg(\exists x)\ P(x)$
  2. $\neg(\forall x)\ P(x) \leftrightarrow (\exists x)\ \neg P(x)$
  3. $(\forall x)\ P(x) \leftrightarrow \neg\ (\exists x)\ \neg P(x)$
  4. $(\exists x)\ P(x) \leftrightarrow \neg(\forall x)\ \neg P(x)$
- Examples
  1. All dogs don't like cats ↔ No dogs like cats
  2. Not all dogs dance ↔ There is a dog that doesn't dance
  3. All dogs sleep ↔ There is no dog that doesn't sleep
  4. There is a dog that talks ↔ Not all dogs can't talk

Simple genealogy KB in FOL

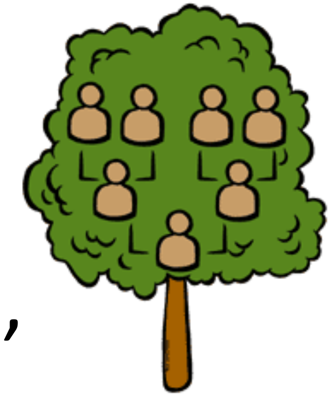
Design a knowledge base using FOL that

- Has facts of immediate family relations, e.g., spouses, parents, etc.
- Defines of more complex relations (ancestors, relatives)
- Detect conflicts, e.g., you are your own parent
- Infers relations, e.g., grandparernt from parent
- Answers queries about relationships between people

How do we approach this?

- Design an initial ontology of types, e.g.
  - e.g., person, man, woman, gender
- Add general individuals to ontology, e.g.
  - gender(male), gender(female)
- Extend ontology be defining relations, e.g.
  - spouse, has_child, has_parent
- Add general constraints to relations, e.g.
  - spouse(X,Y) => ~ X = Y
  - spouse(X,Y) => person(X), person(Y)
- Add FOL sentences for inference, e.g.
  - spouse(X,Y) ⇔ spouse(Y,X)
  - man(X) ⇔ person(X) ∧ has_gender(X, male)

Simple genealogy KB in FOL

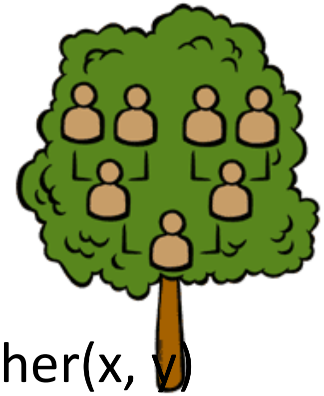


- Has facts of immediate family relations, e.g., spouses, parents, etc.
- Has definitions of more complex relations (ancestors, relatives)
- Can detect conflicts, e.g., you are your own parent
- Can infer relations, e.g., grandparernt from parent
- Can answer queries about relationships between people

Example: A simple genealogy KB by FOL

- Predicates:
  - parent(x, y), child(x, y), father(x, y), daughter(x, y), etc.
  - spouse(x, y), husband(x, y), wife(x,y)
  - ancestor(x, y), descendant(x, y)
  - male(x), female(y)
  - relative(x, y)
- Facts:
  - husband(Joe, Mary), son(Fred, Joe)
  - spouse(John, Nancy), male(John), son(Mark, Nancy)
  - father(Jack, Nancy), daughter(Linda, Jack)
  - daughter(Liz, Linda)
  - etc.

Example Axioms



$(\forall x,y)$ has_parent(x, y) $\leftrightarrow$ has_child (y, x)

$(\forall x,y)$ father(x, y) $\leftrightarrow$ parent(x, y) $\wedge$ male(x) ;similar for mother(x, y)

$(\forall x,y)$ daughter(x, y) $\leftrightarrow$ child(x, y) $\wedge$ female(x) ;similar for son(x, y)

$(\forall x,y)$ husband(x, y) $\leftrightarrow$ spouse(x, y) $\wedge$ male(x) ;similar for wife(x, y)

$(\forall x,y)$ spouse(x, y) $\leftrightarrow$ spouse(y, x) ;spouse relation is symmetric

$(\forall x,y)$ parent(x, y) $\rightarrow$ ancestor(x, y)

$(\forall x,y)(\exists z)$ parent(x, z) $\wedge$ ancestor(z, y) $\rightarrow$ ancestor(x, y)

$(\forall x,y)$ descendant(x, y) $\leftrightarrow$ ancestor(y, x)

$(\forall x,y)(\exists z)$ ancestor(z, x) $\wedge$ ancestor(z, y) $\rightarrow$ relative(x, y)

$(\forall x,y)$ spouse(x, y) $\rightarrow$ relative(x, y) ;related by marriage

$(\forall x,y)(\exists z)$ relative(z, x) $\wedge$ relative(z, y) $\rightarrow$ relative(x, y) ;transitive

$(\forall x,y)$ relative(x, y) $\leftrightarrow$ relative(y, x) ;symmetric

- Rules for genealogical relations

  $(\forall x,y)$ parent(x, y) $\leftrightarrow$ child (y, x)

  $(\forall x,y)$ father(x, y) $\leftrightarrow$ parent(x, y) $\wedge$ male(x) ;similarly for mother(x, y)

  $(\forall x,y)$ daughter(x, y) $\leftrightarrow$ child(x, y) $\wedge$ female(x) ;similarly for son(x, y)

  $(\forall x,y)$ husband(x, y) $\leftrightarrow$ spouse(x, y) $\wedge$ male(x) ;similarly for wife(x, y)

  $(\forall x,y)$ spouse(x, y) $\leftrightarrow$ spouse(y, x) ;spouse relation is symmetric

  $(\forall x,y)$ parent(x, y) $\rightarrow$ ancestor(x, y)

  $(\forall x,y)(\exists z)$ parent(x, z) $\wedge$ ancestor(z, y) $\rightarrow$ ancestor(x, y)

  $(\forall x,y)$ descendant(x, y) $\leftrightarrow$ ancestor(y, x)

  $(\forall x,y)(\exists z)$ ancestor(z, x) $\wedge$ ancestor(z, y) $\rightarrow$ relative(x, y)

  ;related by common ancestry

  $(\forall x,y)$ spouse(x, y) $\rightarrow$ relative(x, y) ;related by marriage

  $(\forall x,y)(\exists z)$ relative(z, x) $\wedge$ relative(z, y) $\rightarrow$ relative(x, y) ;transitive

  $(\forall x,y)$ relative(x, y) $\leftrightarrow$ relative(y, x) ;symmetric

- Queries
  - ancestor(Jack, Fred) ; the answer is yes
  - relative(Liz, Joe) ; the answer is yes
  - relative(Nancy, Matthew) ;no answer, no under closed world assumption
  - $(\exists z)$ ancestor(z, Fred) $\wedge$ ancestor(z, Liz)

Axioms, definitions and theorems

- Axioms: facts and rules that capture the (important) facts and concepts about a domain; axioms can be used to prove theorems
  - Mathematicians dislike unnecessary (dependent) axioms, i.e. ones that can be derived from others
  - Dependent axioms can make reasoning faster, however
  - Choosing a good set of axioms is a design problem
- A definition of a predicate is of the form "$p(X) \leftrightarrow \ldots$" and can be decomposed into two parts
  - Necessary description: "$p(x) \rightarrow \ldots$"
  - Sufficient description "$p(x) \leftarrow \ldots$"
  - Some concepts have definitions (triangle) and some do not (person)

More on definitions

Example: define father(x, y) by parent(x, y) and male(x)

- parent(x, y) is a necessary (but not sufficient) description of father(x, y)

  $\text{father}(x, y) \rightarrow \text{parent}(x, y)$

- parent(x, y) ^ male(x) ^ age(x, 35) is a sufficient (but not necessary) description of father(x, y):

  $\text{father}(x, y) \leftarrow \text{parent}(x, y) \wedge \text{male}(x) \wedge \text{age}(x, 35)$

- parent(x, y) ^ male(x) is a necessary and sufficient description of father(x, y)

  $\text{parent}(x, y) \wedge \text{male}(x) \leftrightarrow \text{father}(x, y)$

Notational differences

- Different symbols for and, or, not, implies, ...
  - $\forall \quad \exists \quad \Rightarrow \quad \Leftrightarrow \quad \wedge \quad \vee \quad \neg \quad \bullet \quad \supset$
  - p v (q ^ r)
  - p + (q * r)
- Prolog

  ```
  cat(X) :- furry(X), meows (X), has(X, claws)
  ```

- Lispy notations

  ```
  (forall ?x (implies (and (furry ?x)
                           (meows ?x)
                           (has ?x claws))
                      (cat ?x)))
  ```

A example of FOL in use

- Semantics of W3C’s semantic web stack (RDF, RDFS, OWL) is defined in FOL
- OWL Full is equivalent to FOL
- Other OWL profiles support a subset of FOL and are more efficient
- However, the semantics of schema.org is only defined in natural language text
- …and Google’s knowledge Graph probably (!) uses probabilities

FOL Summary

- First order logic (FOL) introduces predicates, functions and quantifiers
- More expressive, but reasoning more complex
  - Reasoning in propositional logic is NP hard, FOL is semi-decidable
- Common AI knowledge representation language
  - Other KR languages (e.g., OWL) are often defined by mapping them to FOL
- FOL variables range over objects
  - HOL variables range over functions, predicates or sentences