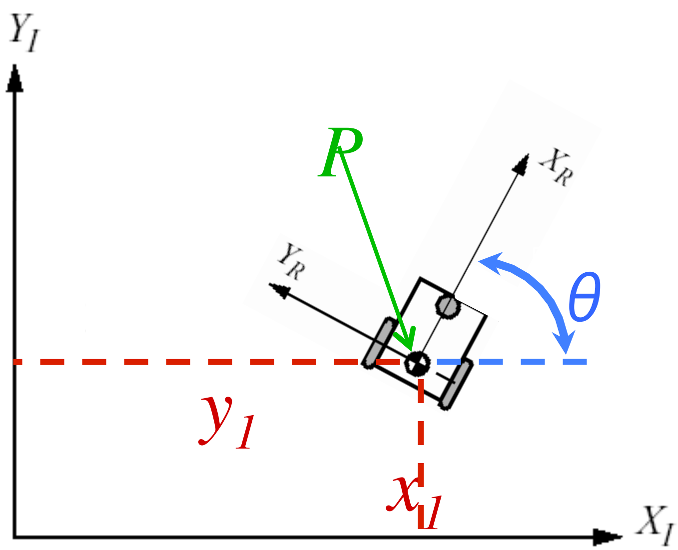
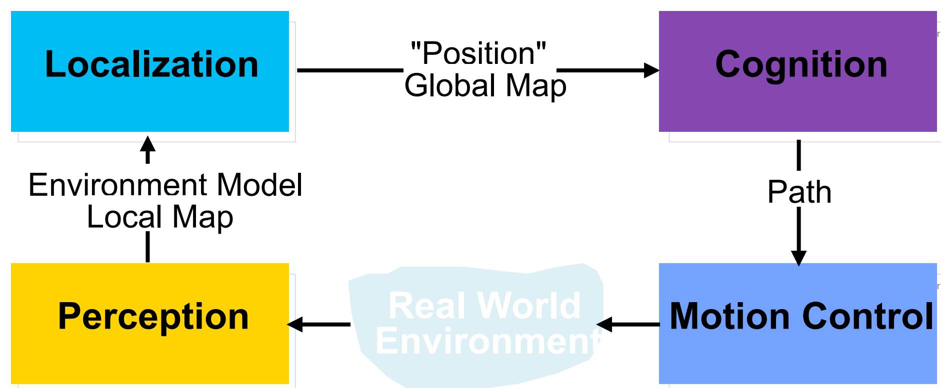
# Kinematics

## *Mobile Kinematics*



# Bookkeeping



- Quiz 2 now visible
- Upcoming
  - Assignment 2: now Friday 11:59pm
    - Workaround for virtualbox bug distributed
    - Nisha's extra hours: tomorrow (Friday) 2-4; tonight?
    - Check submission link **today**
  - **Projects: groups schedule a meeting with me**
- Today
  - General notes on project progress
  - Mobile kinematics and frames of reference
- Reading: Brush up on CB section 1 (from last week)

# Project Progress



- What should milestones look like?
  - Demos
  - Writeups
  - Code
  - Images
- What should they contain?
- What should each group be doing?
- What's here and what's missing?

# Kinematics



- Kinematics:
  - Geometrically possible motion of a body or system of bodies without consideration of the causes and effects of the motions
    - What's an example of what's not a possible motion?

- For mobile robots: position and orientation
  - Kinematics:
    - I moved this way. Where am I and where am I pointed?
  - Inverse Kinematics (IK):
    - I'm here, pointed this way. What motions got me there?
    - I want to be here pointed this way. What motions should I make?
- Position and orientation wrt. an arbitrary initial frame

# Mobile Robot Kinematics

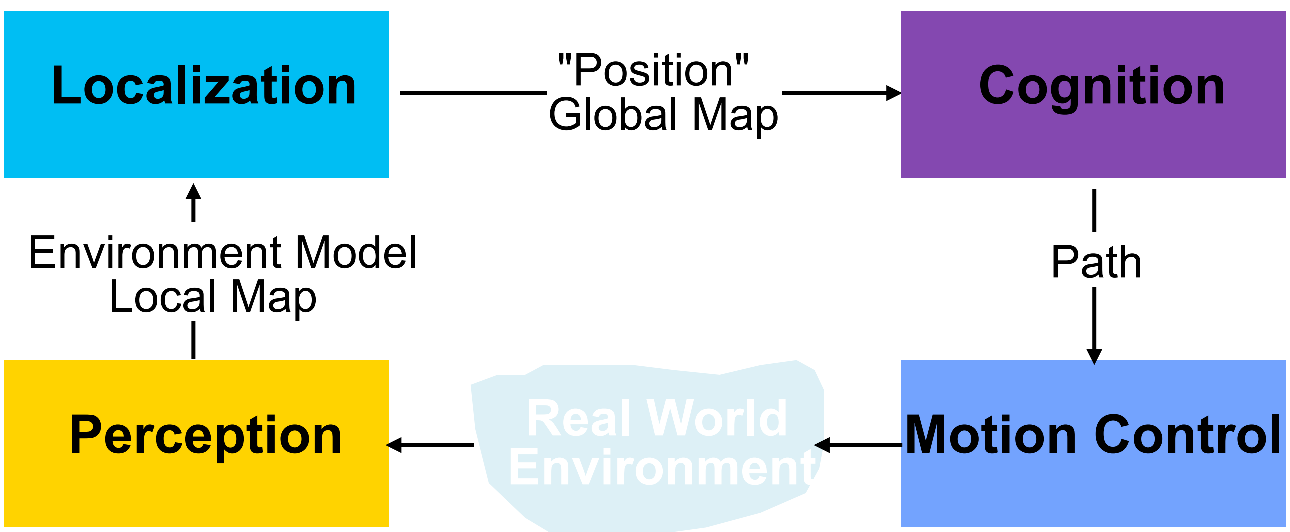


- Goals
  - Find description of mechanical behavior for…
    - Design purposes – how do we design it to do what we need?
    - Control purposes – how do we then get it to do that?
  - Mobile robots can move with respect to environment
    - No direct way to measure the robot's position
    - Position must be integrated over time
    - Leads to inaccuracies of the position (motion) estimate
  - Understanding mobile robot motion starts with **understanding constraints on the robot's mobility.**
- Understanding motion without forces is kinematics.

# Wheeled Motion Control



- Requirements for understanding/controlling motion:
  - Kinematic / dynamic model of the robot.
  - Model of interaction between the wheel and the ground
  - Definition of required motion
    - What speed and position controls are there? Are possible?
  - A control policy that satisfies the requirements



# Mobile Position & Orientation



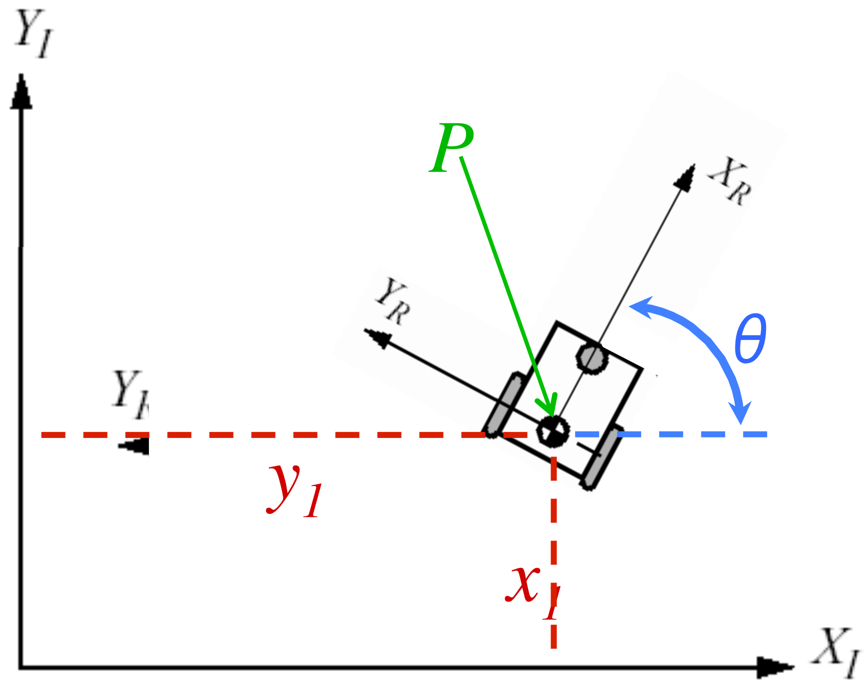
Frames of reference:

$\{X_I, Y_I\}$: Global
$\{X_R, Y_R\}$: Robot

Robot: point $P$

Position (of P):
$\{x_{I,1}, y_{I,1}\}$

Heading:
$\{\theta\}$: I $\angle$ R

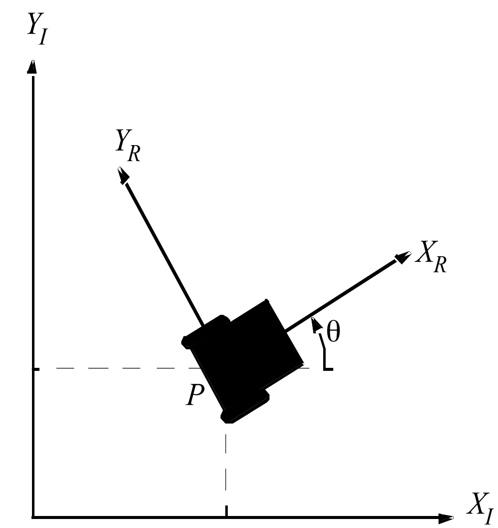



$$\xi_I = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix}$$

# Mapping Between Frames

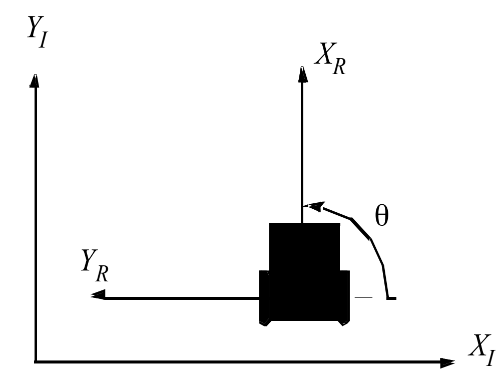


- Representing robot within an arbitrary initial frame
  - Initial frame: $\{X_I, Y_I\}$
  - Robot frame: $\{X_R, Y_R\}$
  - Robot: $\xi_I = \begin{bmatrix} x & y & \theta \end{bmatrix}^T$




- Goal
  - Map motions from global reference frame to local reference frame (and sometimes vice versa)

$$\{X_I, Y_I\} \longrightarrow \{X_R, Y_R\}$$



# Mapping Between Frames

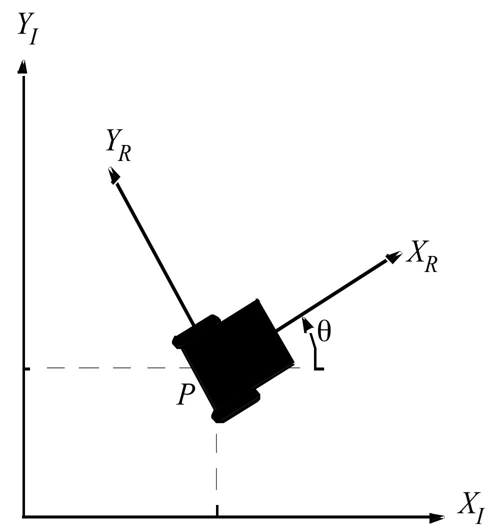


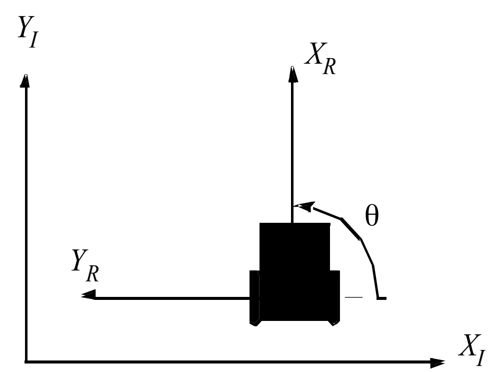
- Global reference frame ←→ local reference frame

$$\{X_I, Y_I\} \leftrightarrow \{X_R, Y_R\}$$

  - Map motion from **axes** of one to **axes** of the other
    - This mapping depends on current pose
- Use *orthogonal reference frame*:

$$R(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



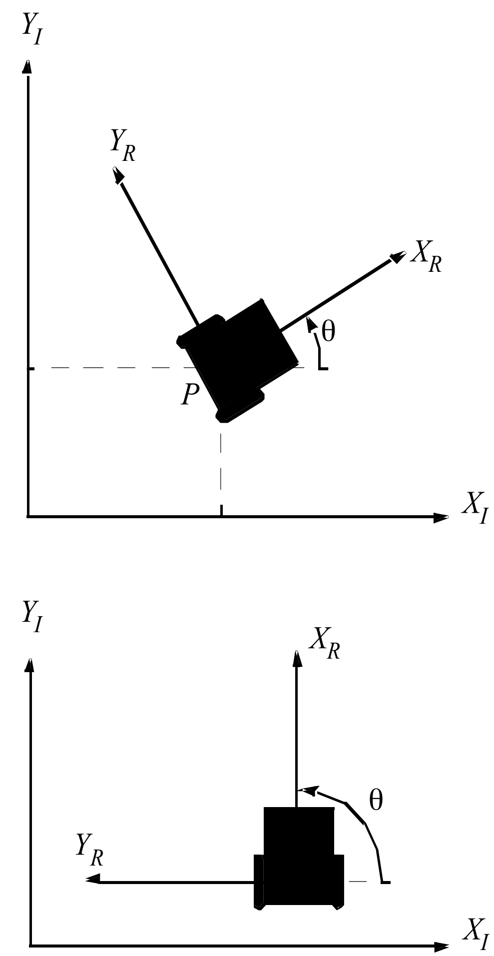
# Mapping Between Frames



- Global reference frame ←→ local reference frame

$$\{X_I, Y_I\} \longleftrightarrow \{X_R, Y_R\}$$

  - Map motion from **axes** of one to **axes** of the other
    - This mapping depends on current pose
- How do you do this mapping?
- How do you perform a rotation in Euclidean spaces?

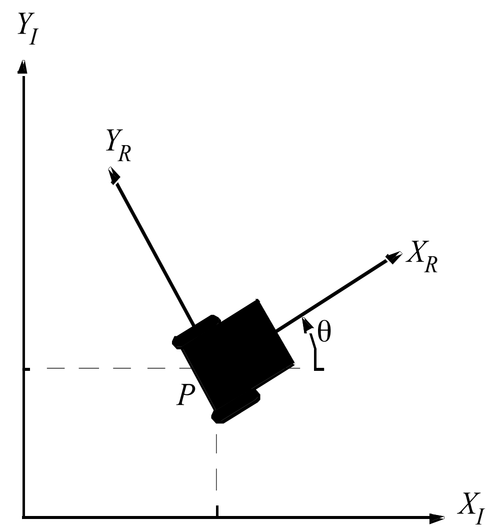


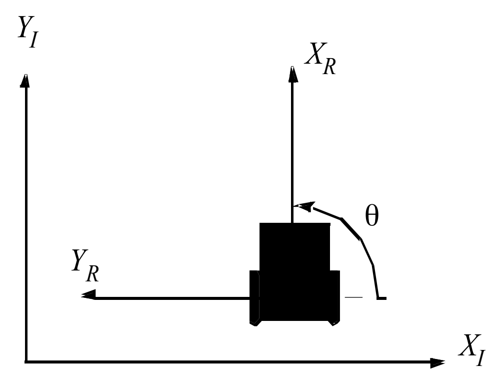
# Mapping Between Frames



- How do you perform a rotation?
- "A **rotation matrix** is a matrix that is used to perform a rotation in Euclidean space."
  - Example: rotation matrix *R*

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$




- Rotates points in the *xy* plane counter-clockwise, through *θ*, around the origin.
- To use *R*, the **position of each point** must be represented by a vector.
- A rotated vector is then obtained with matrix multiplication.

# Orthogonal Rotation Matrix



- This mapping function is called

  $R(\theta)\,\dot{\xi}_I$

  because it depends on $\theta$.

$$R(\theta)=\begin{bmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}$$

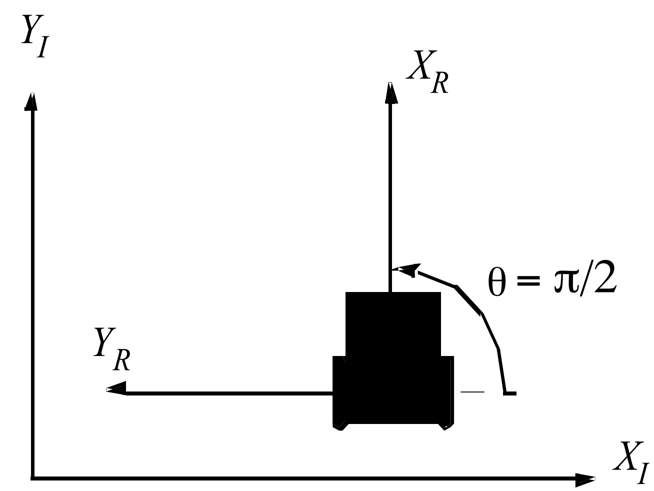
orthogonal rotation matrix

$$\dot{\xi}_R = R(\pi/2)\,\dot{\xi}_I$$

$$R(\pi/2)=\begin{bmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}$$

Why?

*because trig!*




cos(pi/2)=0
sin(pi/2) =1
tan(pi/2) =infty
cot(pi/2) =0
csc(pi/2)=1
sec(pi/2)=infty

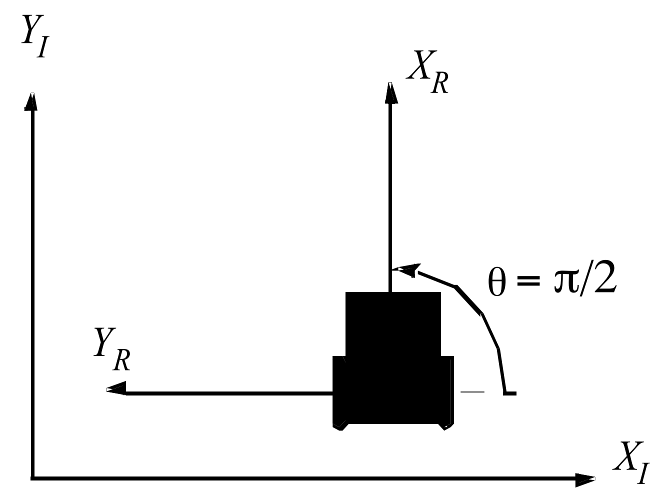
# Velocity Vector



- Given some velocity in *I*:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix}$$

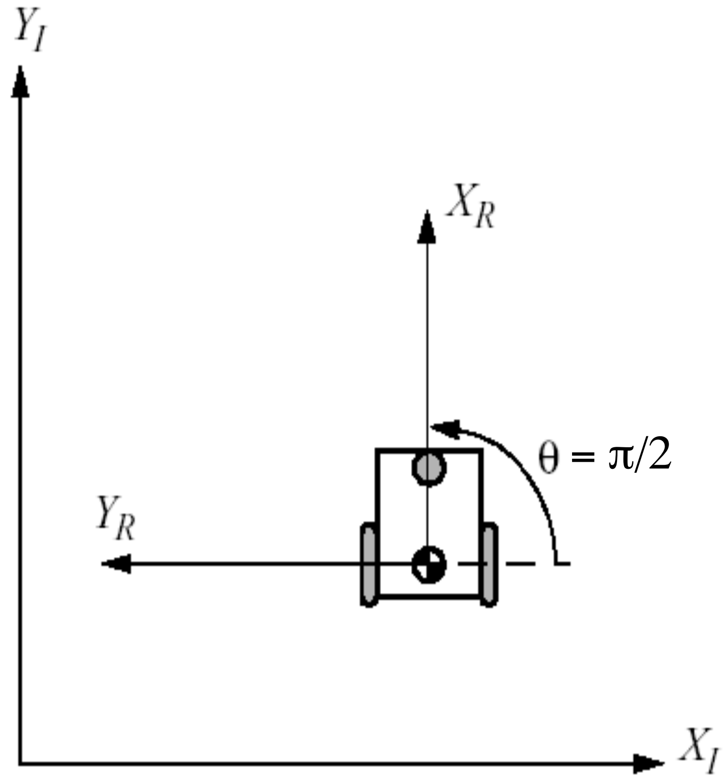
- We can compute motion along $X_R$ and $Y_R$.
  - Motion along $X_R = \dot{y}$
  - Motion along $Y_R = -\dot{x}$



# Example, cont'd



$$R(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

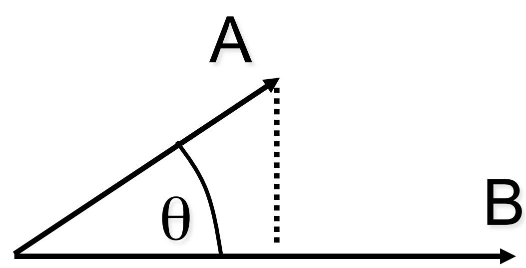



$$\dot{\xi}_R = R(\frac{\pi}{2})\dot{\xi}_I = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{y} \\ -\dot{x} \\ \dot{\theta} \end{bmatrix}$$

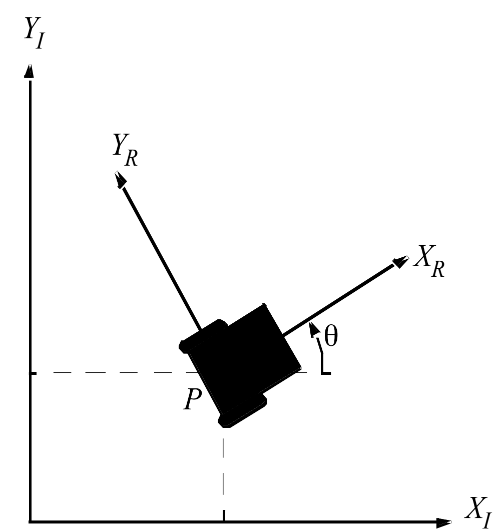
# Mapping Between Frames



$$\dot{\xi}_R = R(\theta)\,\dot{\xi}_I = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{x}\cos\theta + \dot{y}\sin\theta \\ -\dot{x}\sin\theta + \dot{y}\cos\theta \\ \dot{\theta} \end{bmatrix}$$




What is $|A|\cos\theta$ ?



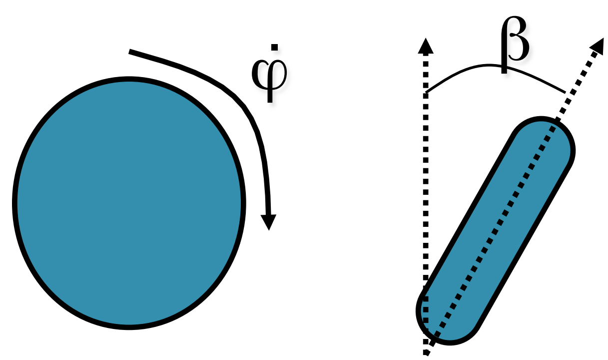

Recall

- $\cos(\pi / 2 - \theta) = \sin(\theta)$
- $\cos(\pi / 2 + \theta) = -\sin(\theta)$

# Kinematics Models



- Goal:
  - Establish speed $\dot{\xi} = [\dot{x} \quad \dot{y} \quad \dot{\theta}]^T$ as a function of the wheel speeds $\dot{\varphi}_i$, steering angles $\beta_i$ , steering speeds $\dot{\beta}_i$ and the geometric parameters of the robot (configuration coordinates)




- $\dot{\varphi}$ measured in radians/sec, so $\dot{\varphi}/2\pi$ is revolutions/sec
- In one revolution wheel translates $2\pi r$ linear units
- Translational velocity is $2\pi r(\dot{\varphi}/2\pi) = r\dot{\varphi}$

# Forward Kinematics Models



- Goal:
  - Establish speed $\dot{\xi} = \begin{bmatrix} \dot{x} & \dot{y} & \dot{\theta} \end{bmatrix}^T$ as a function of the wheel speeds $\dot{\varphi}_i$, steering angles $\beta_i$, steering speeds $\dot{\beta}_i$ and the geometric parameters of the robot (configuration coordinates)

- Forward kinematics:

"If I do this, what will happen?"

$$\dot{\xi} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = f(\dot{\varphi}_1, \cdots \dot{\varphi}_n, \beta_1, \cdots \beta_m, \dot{\beta}_1, \cdots \dot{\beta}_m)$$

# Inverse Kinematics Models



- Goal:
  - Establish speed $\dot{\xi} = \begin{bmatrix} \dot{x} & \dot{y} & \dot{\theta} \end{bmatrix}^T$ as a function of the wheel speeds $\dot{\varphi}_i$, steering angles $\beta_i$, steering speeds $\dot{\beta}_i$ and the geometric parameters of the robot (configuration coordinates)

- Inverse kinematics:

"If I want this to happen, what should I do?"

$$\begin{bmatrix} \dot{\varphi}_1 & \cdots & \dot{\varphi}_n & \beta_1 & \cdots & \beta_m & \dot{\beta}_1 & \cdots & \dot{\beta}_m \end{bmatrix}^T = f(\dot{x}, \dot{y}, \dot{\theta})$$

# Differential Drive Model

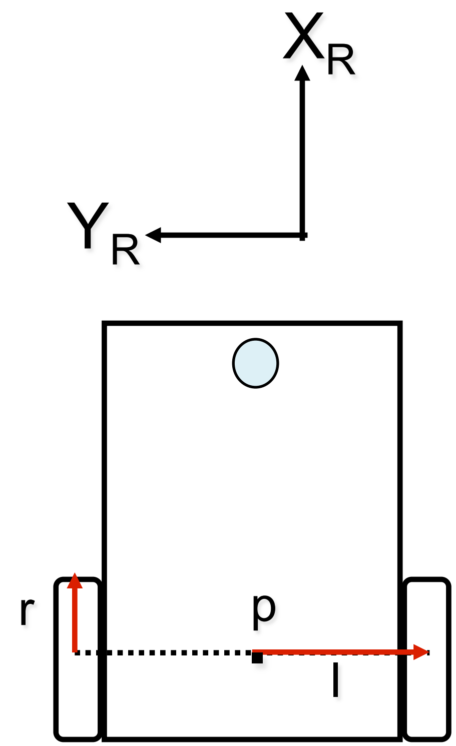


- The robot has:
  - Two wheels - radius *r*
  - Point P centered between wheels
  - Each wheel is distance l ($\ell$) from P
  - Wheels have rotational velocity $\dot{\varphi}_1$ and $\dot{\varphi}_2$
- Forward kinematic model

$$\dot{\xi}_I = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = f(\ell, r, \theta, \dot{\varphi}_1, \dot{\varphi}_2)$$

- Mapping from global to local is

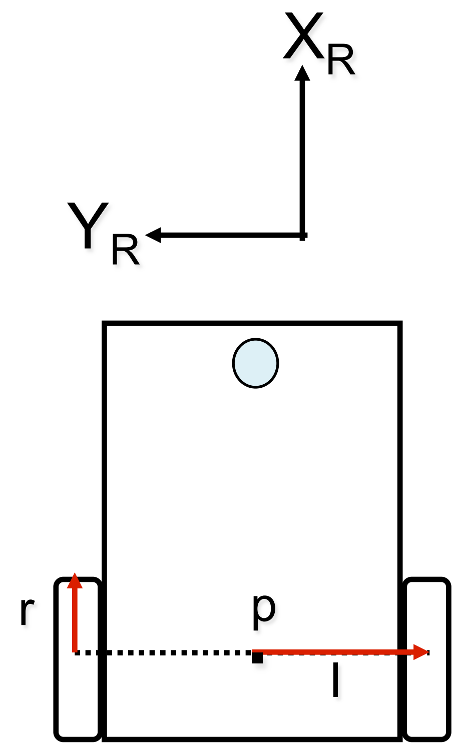
$$\dot{\xi}_R = R(\theta)\,\dot{\xi}_I \text{ , so } \dot{\xi}_I = R^{-1}(\theta)\,\dot{\xi}_R$$



# Differential Drive (cont.)



- Since $\dot{\xi}_R = R(\theta)\,\dot{\xi}_I$ , $\dot{\xi}_I = R^{-1}(\theta)\,\dot{\xi}_R$
- Compute how wheel speeds influence $\dot{\xi}_R$
- Translate to $\dot{\xi}_I$ via $R^{-1}(\theta)$
- Contribution to translation along $X_R$
- If one wheel spins and the other is still:
  - P will move at half the translational velocity of the wheel: $1/2r\,\dot{\varphi}_1$ or $1/2r\,\dot{\varphi}_2$
  - Sum these for both wheels spinning
    - $\dot{X}_R = 1/2r\dot{\varphi}_1 + 1/2r\dot{\varphi}_2$

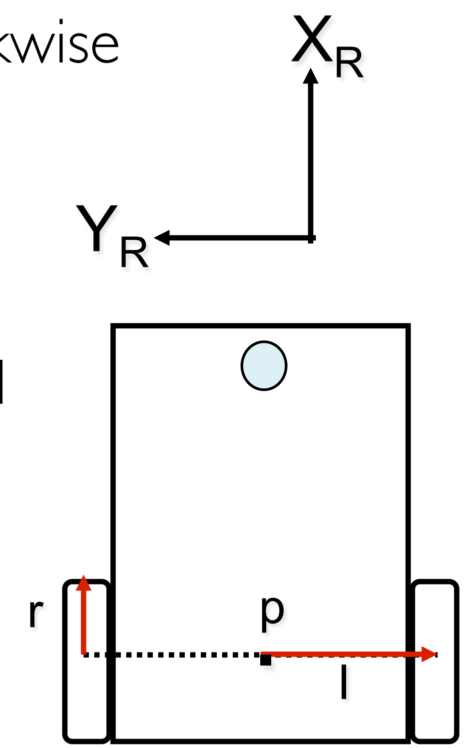



- What if they spin in opposite directions? Same direction?

# Differential Drive (cont.)



- Wheel rotation never contributes to $Y_R$. Why?
- What about $\theta$?
  - Wheel 1 spin makes robot rotate counterclockwise
  - Pivot around wheel 2 (left wheel)
  - Translational velocity is $r\dot{\varphi}$
  - Traces circle with radius 2l
  - Rotational velocity $2\pi * r\dot{\varphi} / (2\pi * 2l) = r\dot{\varphi} / 2l$
  - Wheel 2 spin makes robot rotate clockwise
  - Sum to get net effect: $\dot{\theta} = (r\dot{\varphi}_1 - r\dot{\varphi}_2) / 2l$



# Differential Drive: The Punchline

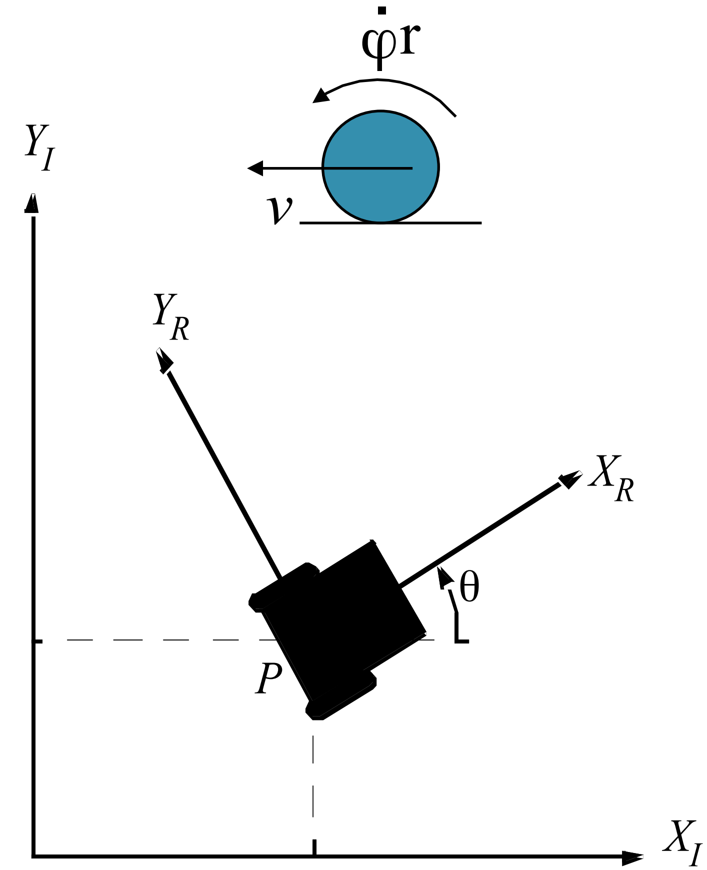


$$\dot{\xi}_I = R^{-1}(\theta)\,\dot{\xi}_R = R^{-1}(\theta)\begin{bmatrix} r(\dot{\varphi}_1 + \dot{\varphi}_2) / 2 \\ 0 \\ r(\dot{\varphi}_1 - \dot{\varphi}_2) / 2l \end{bmatrix}$$

# Wheel Constraints: Assumptions



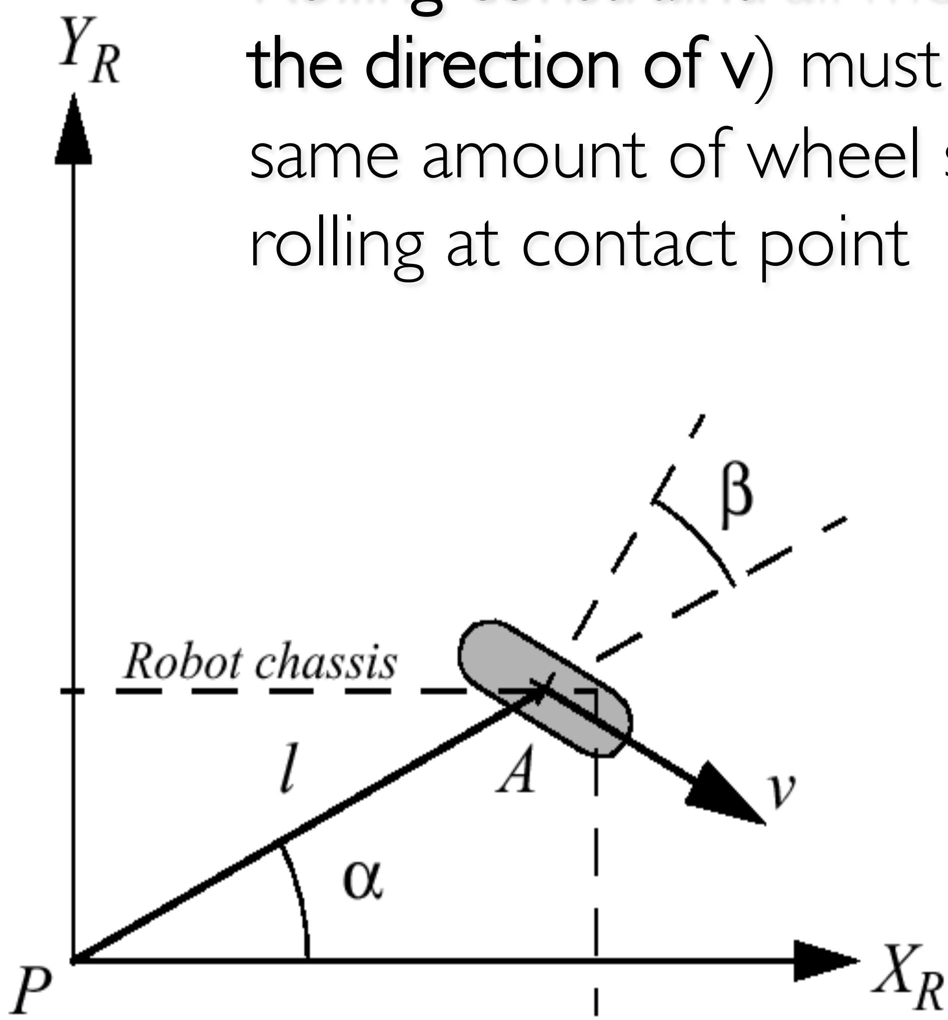
- Movement is on a horizontal plane
- Wheels:
  - Make point contact
  - Are not deformable
  - Are connected to rigid chassis
  - Have steering axes orthogonal to surface being moved on
- Pure rolling
- No slipping, skidding or sliding
- No friction in rotation around contact point

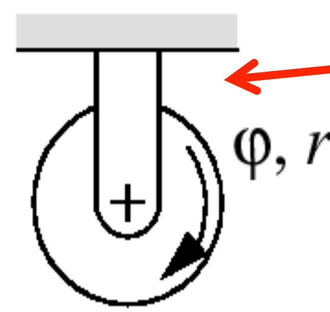


# Wheels: Rolling Constraint



**Rolling constraint:** all motion along wheel plane (**in the direction of v**) must be accompanied by the same amount of wheel spin so that there is pure rolling at contact point



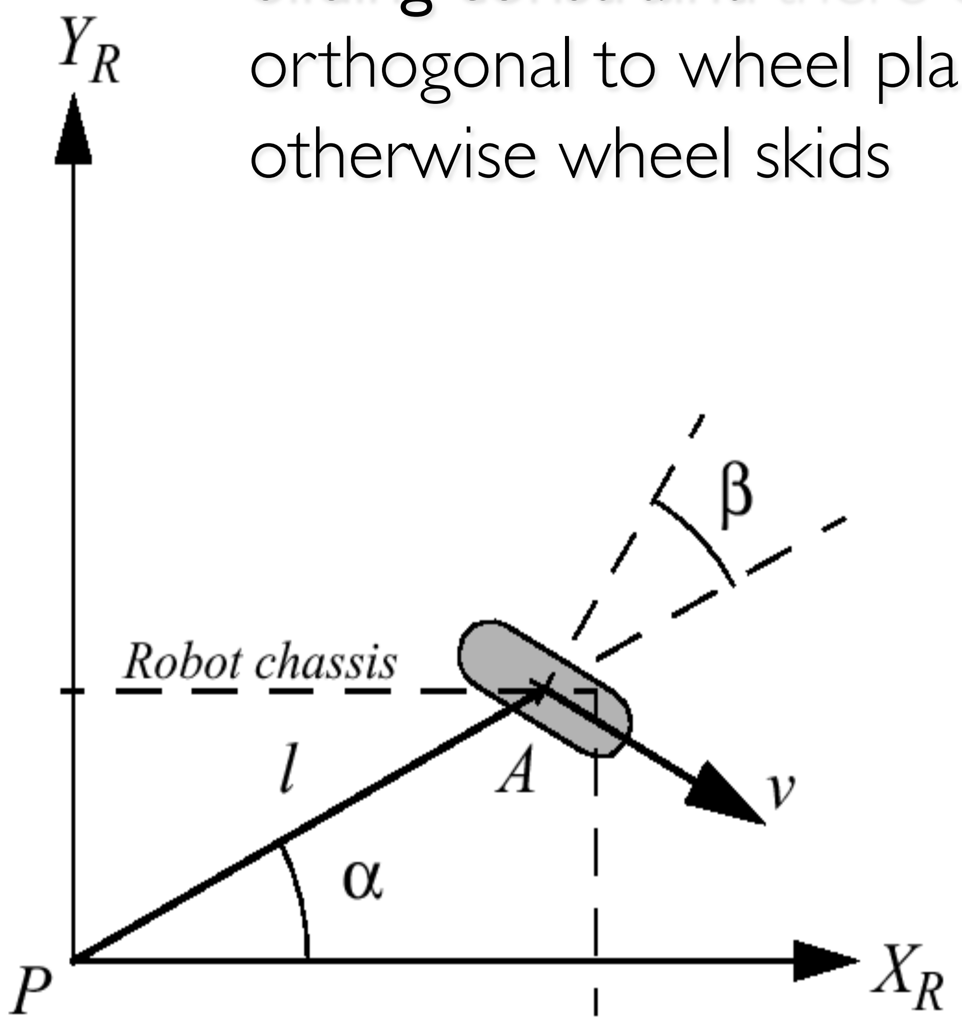



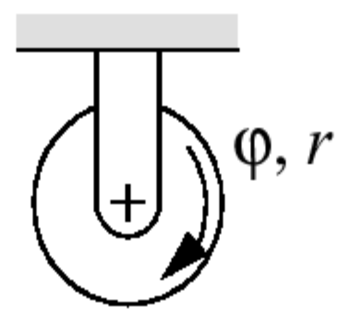

Now we discuss *fixed wheel A*

# Wheels: Sliding Constraint



**Sliding constraint**: there can be no motion orthogonal to wheel plane (perpendicular to v), otherwise wheel skids

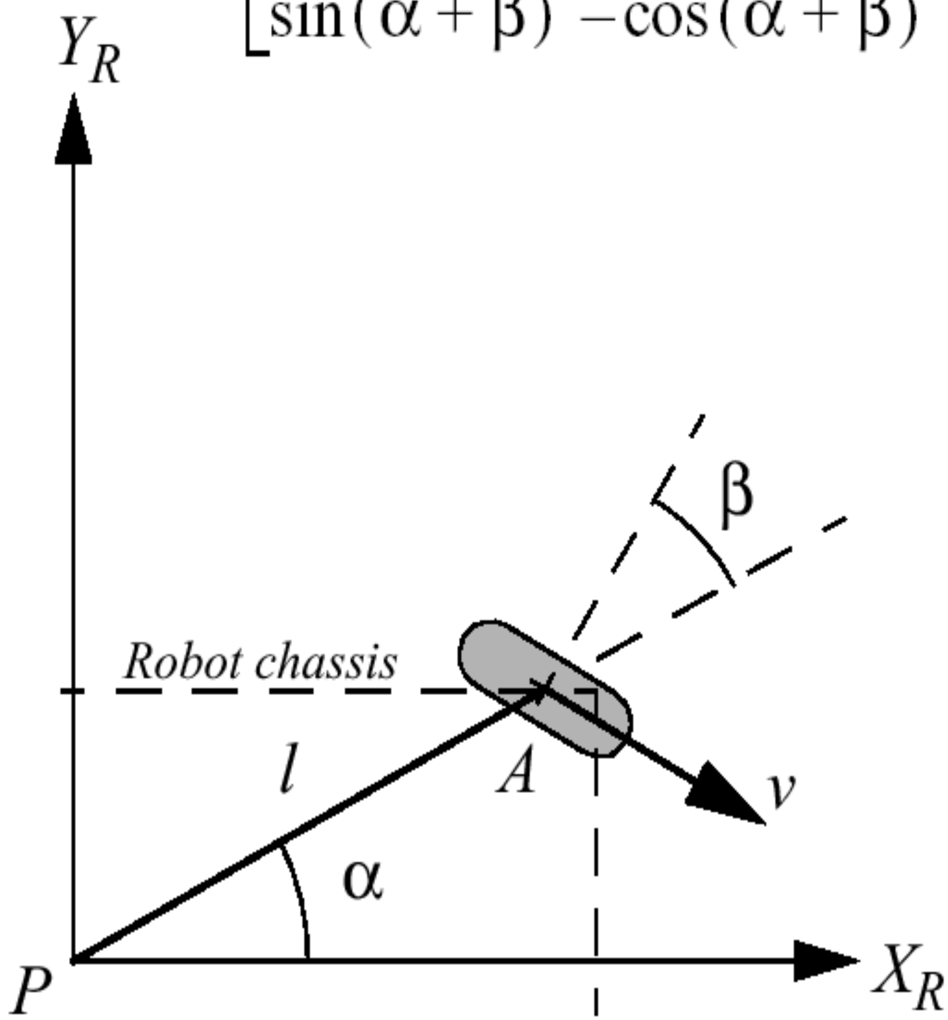


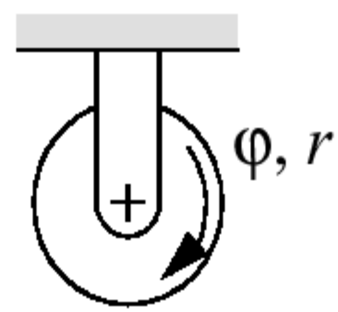



# Wheels: Round Constraint



$$\left[\sin(\alpha+\beta)\ \ -\cos(\alpha+\beta)\ \ (-l)\cos\beta\right] R(\theta)\dot{\xi}_I - r\dot{\varphi} = 0$$

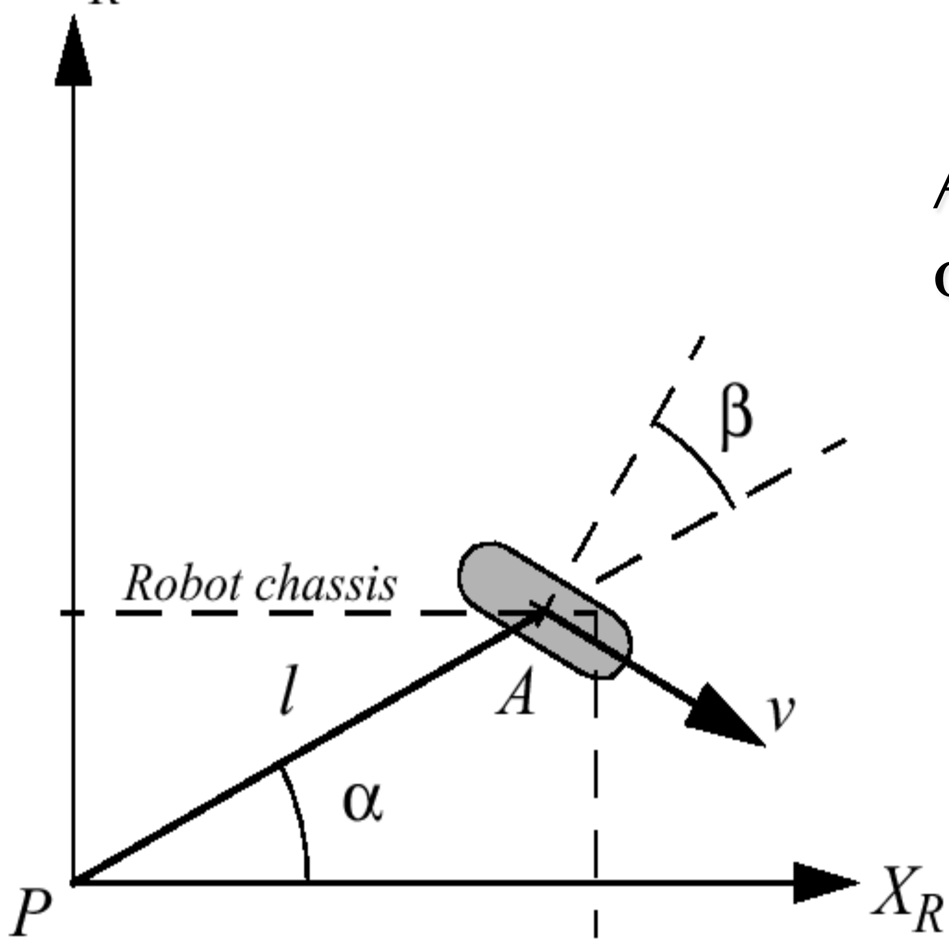


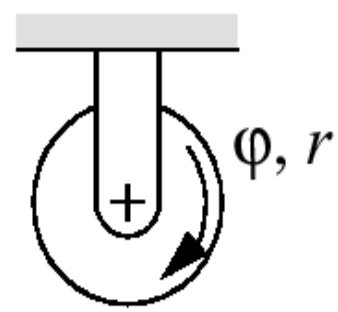
# Round Constraint (2)



$$\begin{bmatrix} \sin(\alpha+\beta) & -\cos(\alpha+\beta) & (-l)\cos\beta \end{bmatrix} R(\theta)\dot{\xi}_I - r\dot{\varphi} = 0$$



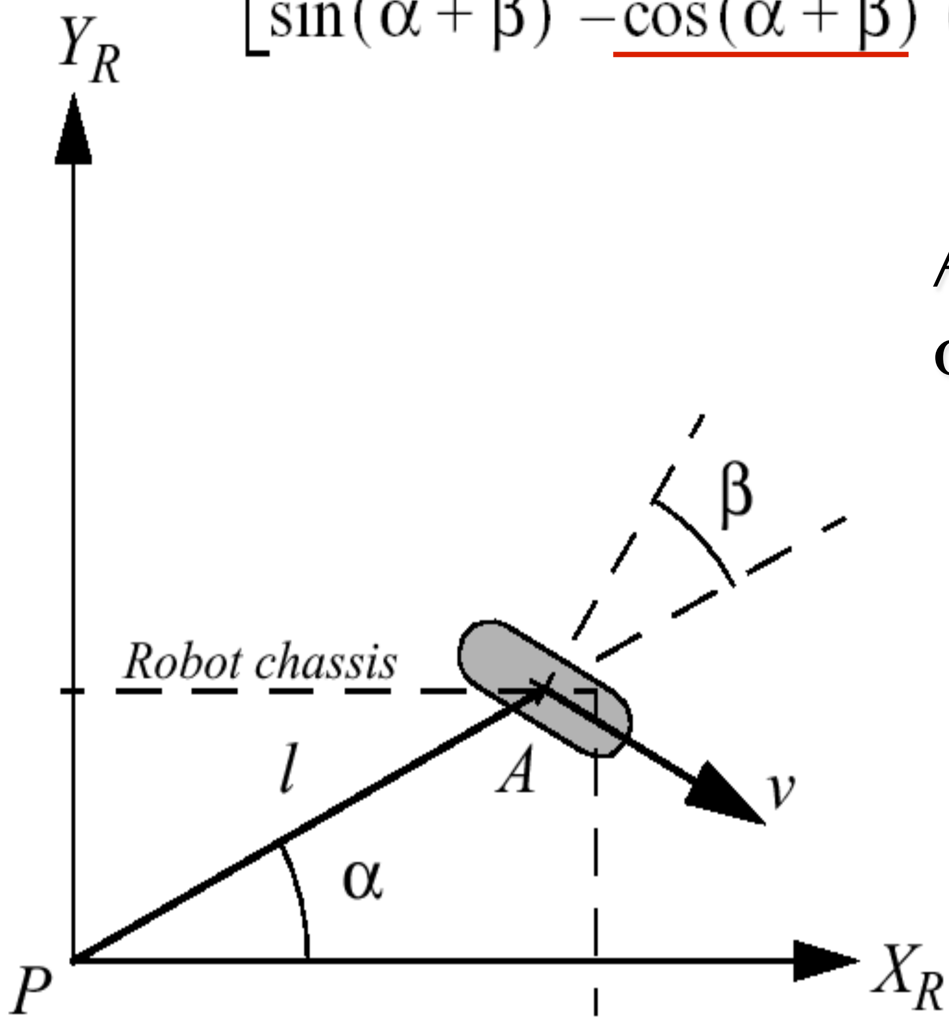

Angle between $X_R$ and v is
$\alpha + \beta - \pi/2$



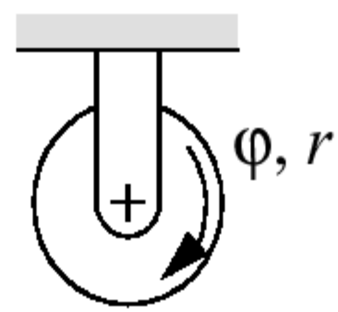
# Round Constraint (3)



$$\begin{bmatrix} \sin(\alpha+\beta) & -\cos(\alpha+\beta) & (-l)\cos\beta \end{bmatrix} R(\theta)\dot{\xi}_I - r\dot{\varphi} = 0$$




Angle between $Y_R$ and v is
$\alpha + \beta - \pi$



# Round Constraint (4)



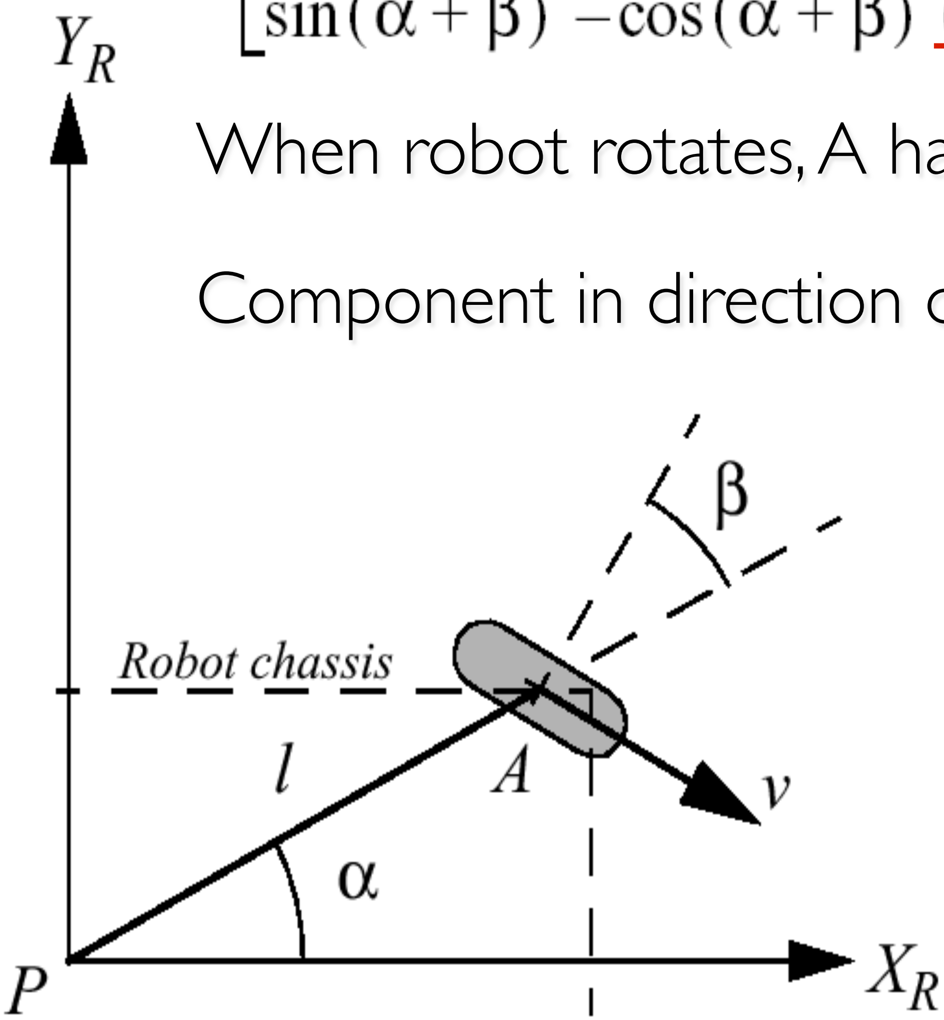
$$\left[\sin(\alpha+\beta)\;\; -\cos(\alpha+\beta)\;\; \underline{(-l)\cos\beta}\right] R(\theta)\dot{\xi}_I - r\dot{\varphi} = 0$$

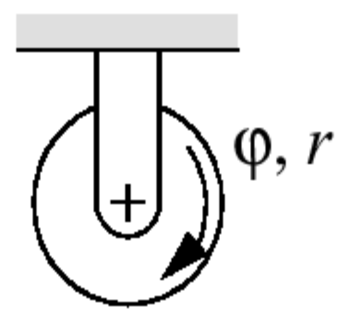
When robot rotates, A has translational velocity $l\theta$.

*Why?*

Component in direction of V is $-l\theta\cos\beta$.

*Why?*

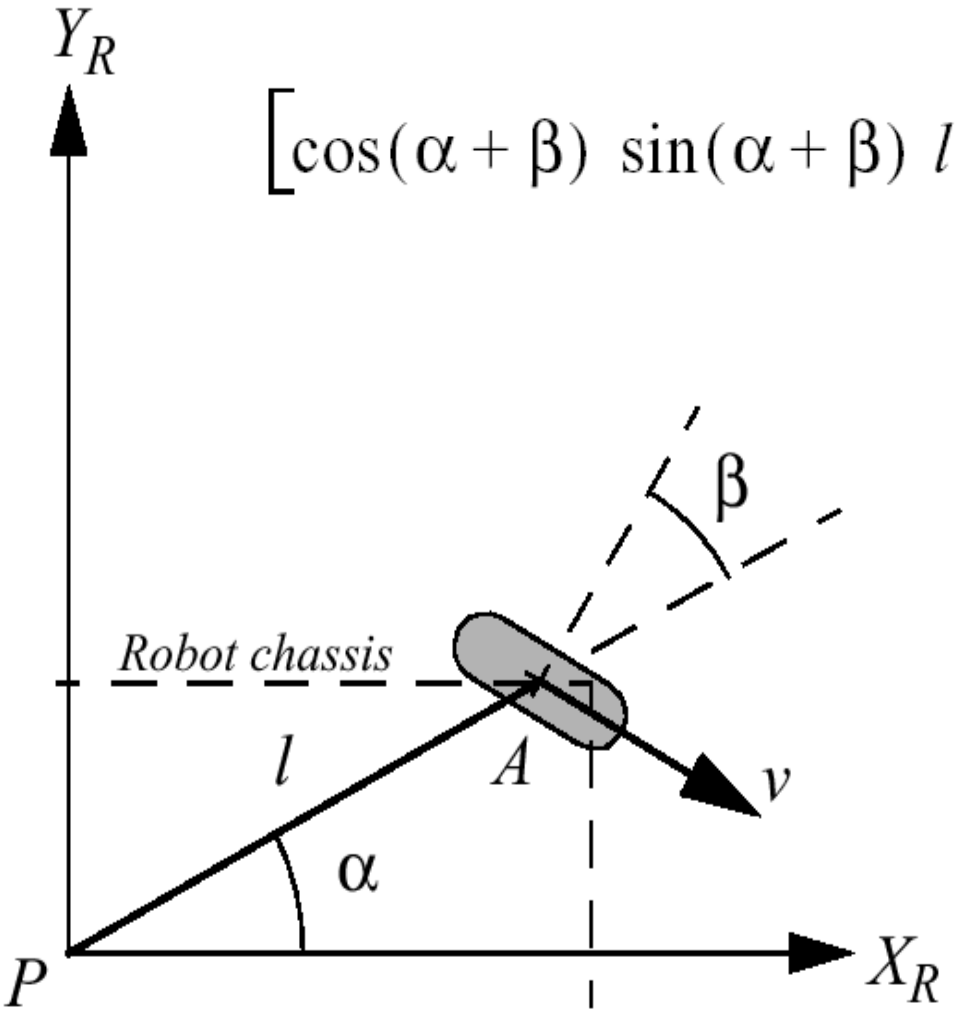


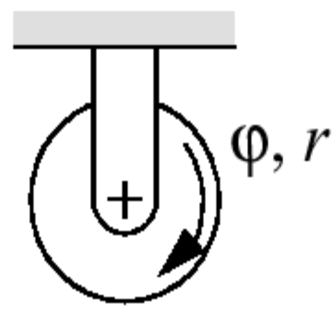



# Sliding Constraint (2)



$$\left[\cos(\alpha+\beta)\ \ \sin(\alpha+\beta)\ \ l\sin\beta\right]R(\theta)\dot{\xi}_I - r\dot{\varphi} = 0$$



# Example



$$\begin{bmatrix}\sin(\alpha+\beta) & -\cos(\alpha+\beta) & (-l)\cos\beta\end{bmatrix} R(\theta)\dot{\xi}_I - r\dot{\varphi} = 0$$
$$\begin{bmatrix}\cos(\alpha+\beta) & \sin(\alpha+\beta) & l\sin\beta\end{bmatrix} R(\theta)\dot{\xi}_I - r\dot{\varphi} = 0$$

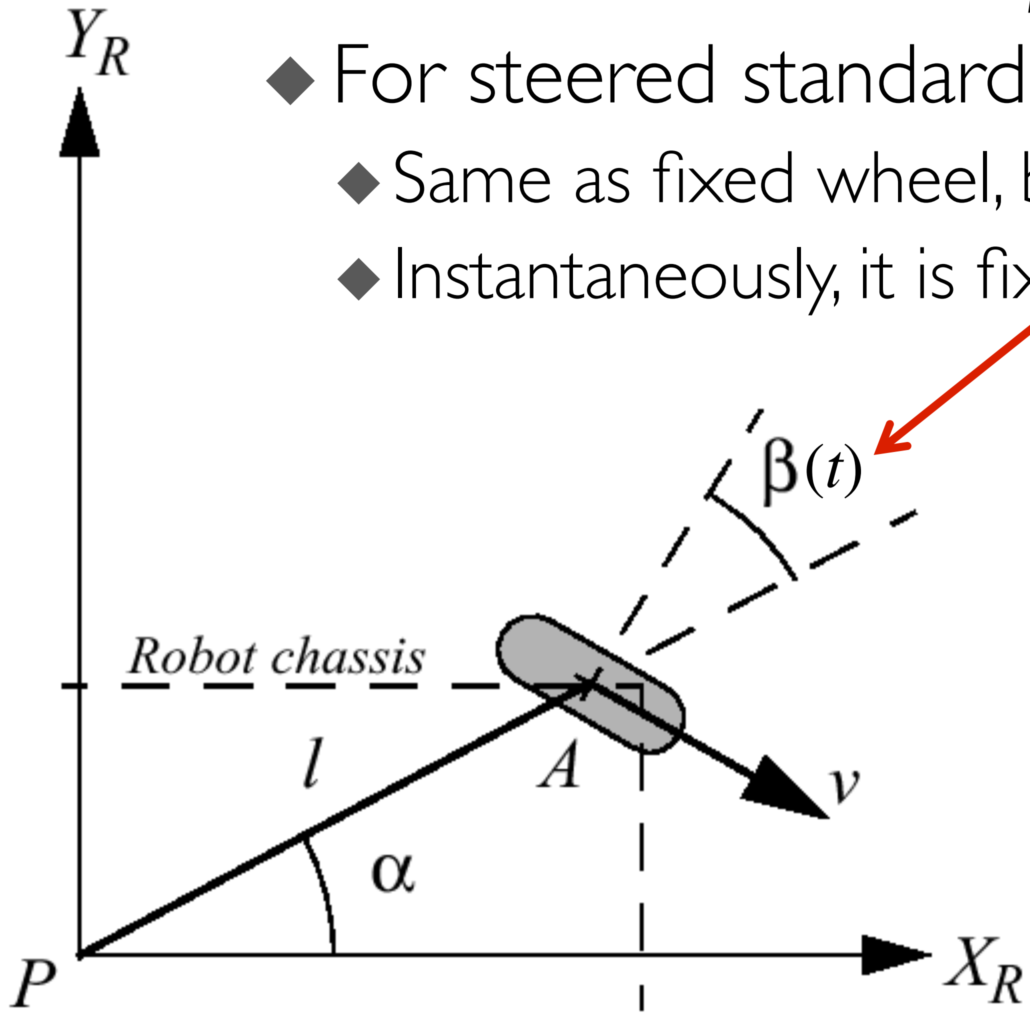
- Suppose that the wheel A is in position such that $\alpha = 0$ and $\beta = 0$
- Puts contact point of wheel on $X_I$, with plane of the wheel oriented parallel to $Y_I$
- If $\theta = 0$, then the sliding constraint reduces to:

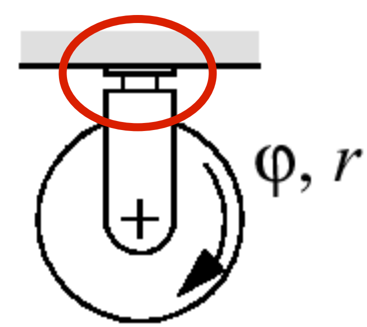
$$\begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{\theta}\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{\theta}\end{bmatrix} = 0$$

# Steered Standard Wheel



- This has all been for *fixed wheels*.
- For steered standard (not offset) wheels:
  - Same as fixed wheel, but β changes over time.
  - Instantaneously, it is fixed.

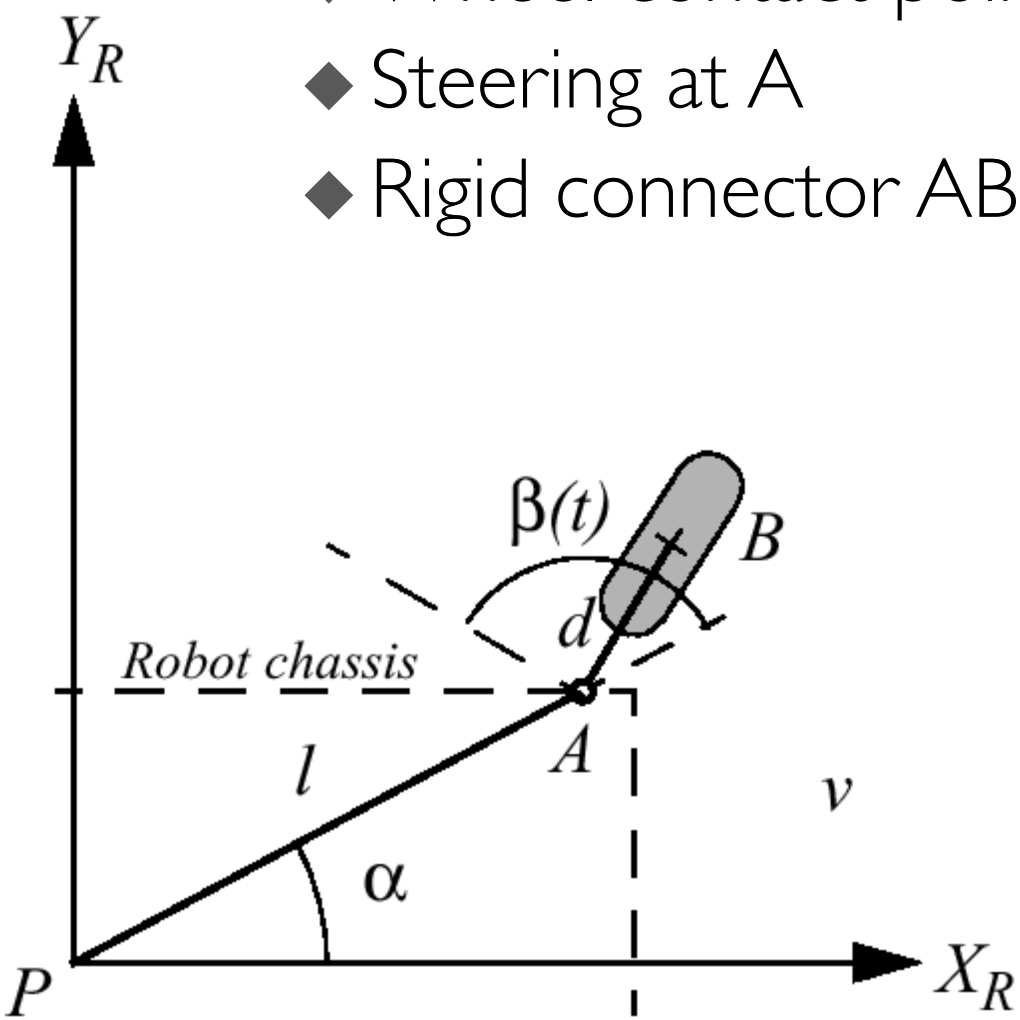


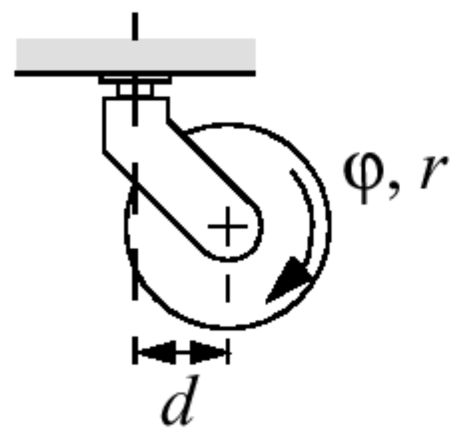



# Castor (Offset) Wheel



- Wheel contact point at B
- Steering at A
- Rigid connector AB



# Not Omnidirectional: Why?



$$\begin{bmatrix} \sin(\alpha+\beta) & -\cos(\alpha+\beta) & (-l)\cos\beta \end{bmatrix} R(\theta)\dot{\xi}_I - r\dot{\varphi} = 0$$

$$\begin{bmatrix} \cos(\alpha+\beta) & \sin(\alpha+\beta) & l\sin\beta \end{bmatrix} R(\theta)\dot{\xi}_I - r\dot{\varphi} = 0$$

- Can constraints be satisfied for ANY $\dot{\xi}_I$?
- How will constraints be used?
- Once again, maneuverability / capability is…?

Inversely proportional to complexity of control

$$Capability \propto \frac{1}{Control\ Complexity}$$