Probability, Decision Theory, and Loss Functions

CMSC 678

UMBC

Some slides adapted from Hamed Pirsiavash

A Terminology Buffet

Classification

Regression

Clustering

the ***task****: what* kind *of problem are you solving?*

A Terminology Buffet

Classification	Fully-supervised
Regression	Semi-supervised
Clustering	Un-supervised
the ***task****: what* kind *of problem are you solving?*	*the* ***data****: amount of human input/number of labeled examples*

A Terminology Buffet

Classification	Fully-supervised	Probabilistic	Neural
Regression	Semi-supervised	Generative	Memory-based
Clustering	Un-supervised	Conditional	Exemplar
		Spectral	…
*the **task**: what* kind *of problem are you solving?*	*the **data**: amount of human input/number of labeled examples*	*the **approach**: how any data are being used*	

Outline

Review+Extension

Probability

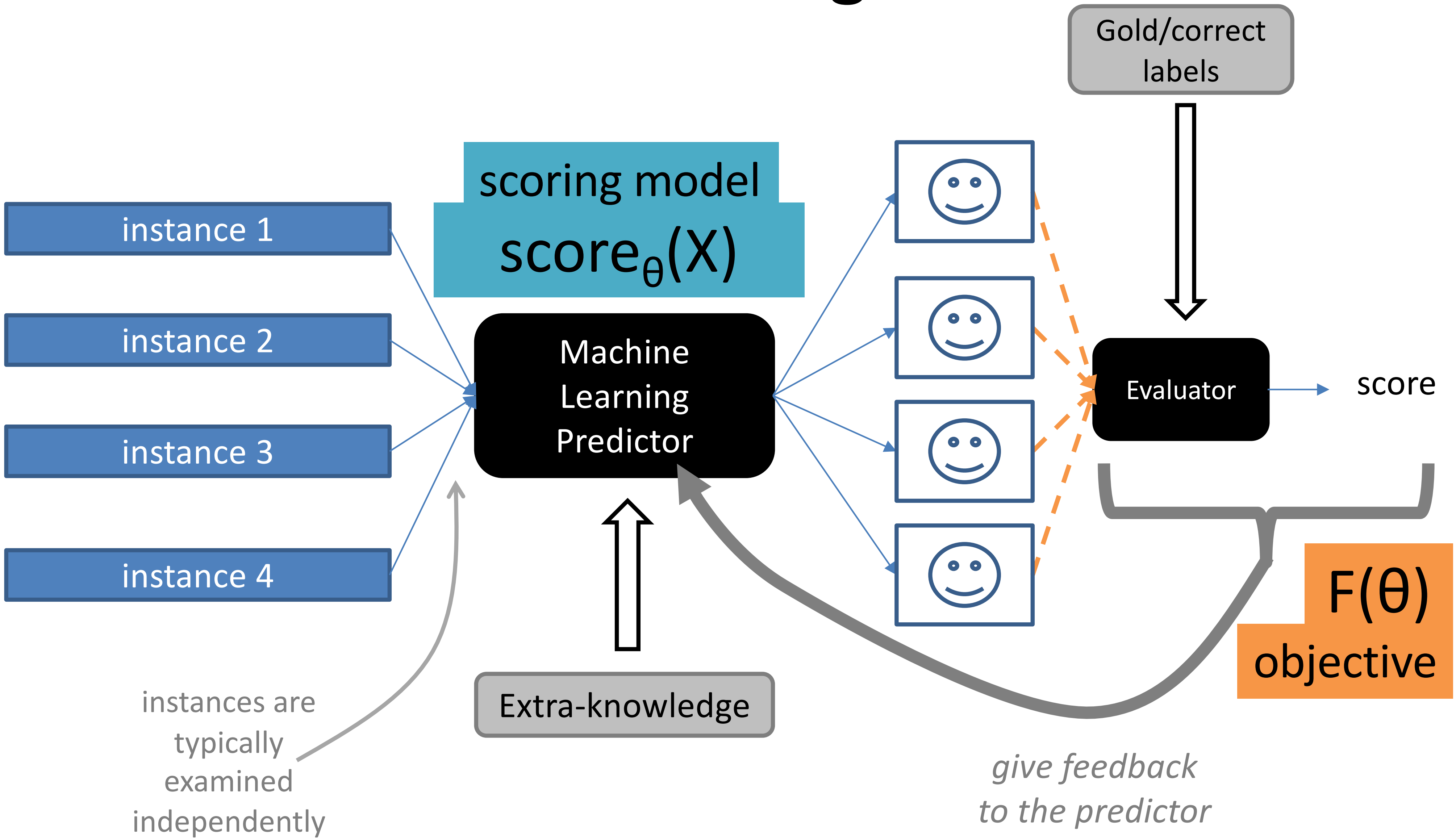
Decision Theory

Loss Functions

What does it mean to learn?

Generalization

Machine Learning Framework: Learning



Model, parameters and hyperparameters

Model: mathematical formulation of system (e.g., classifier)

Parameters: primary "knobs" of the model that are set by a learning algorithm

Hyperparameter: secondary "knobs"

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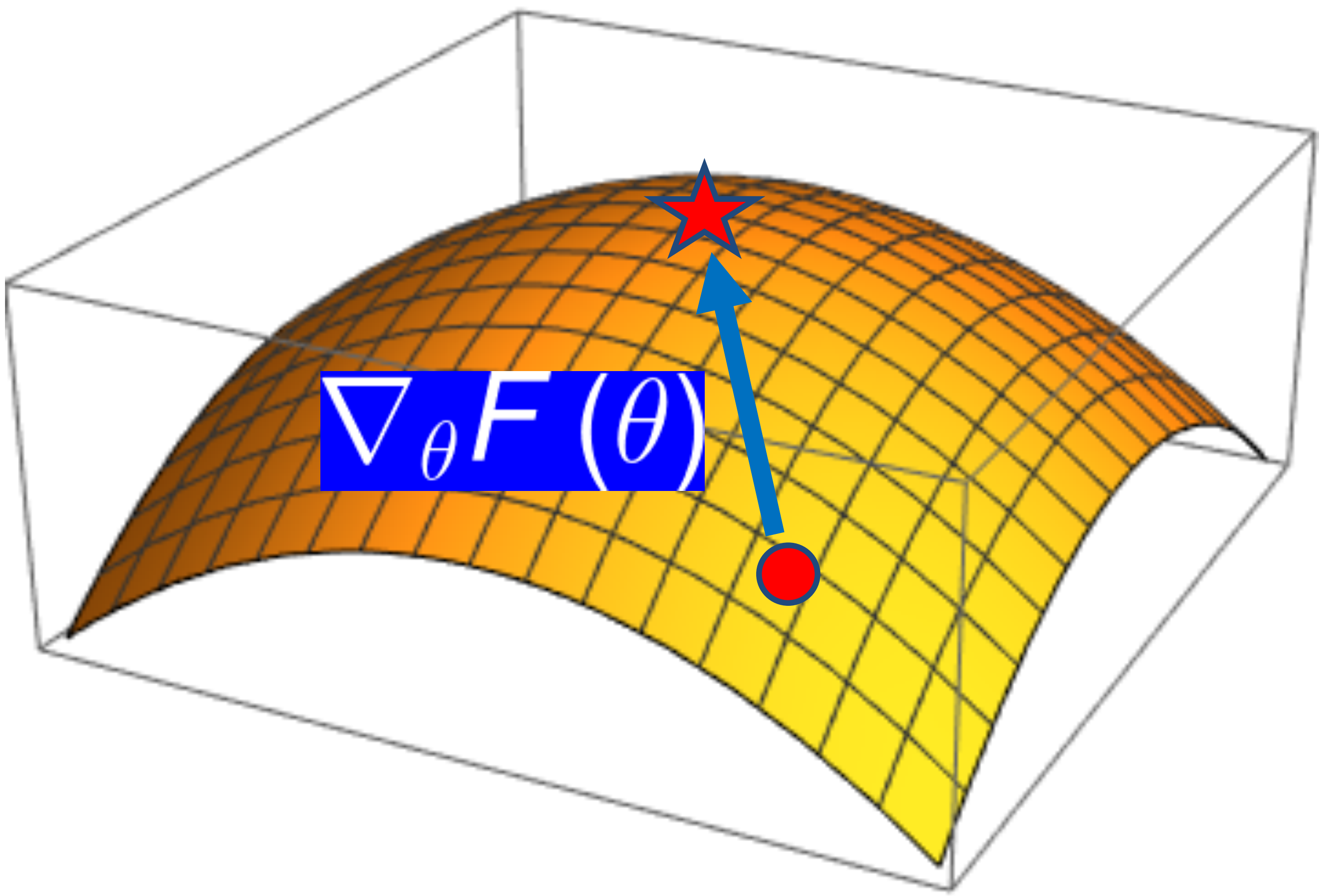
Gradient Ascent

$$\arg\max_{\theta} \; F(\theta)$$



General ML Consideration: Inductive Bias

What do we know *before* we see the data, and how does that influence our modeling decisions?

Courtesy Hamed Pirsiavash

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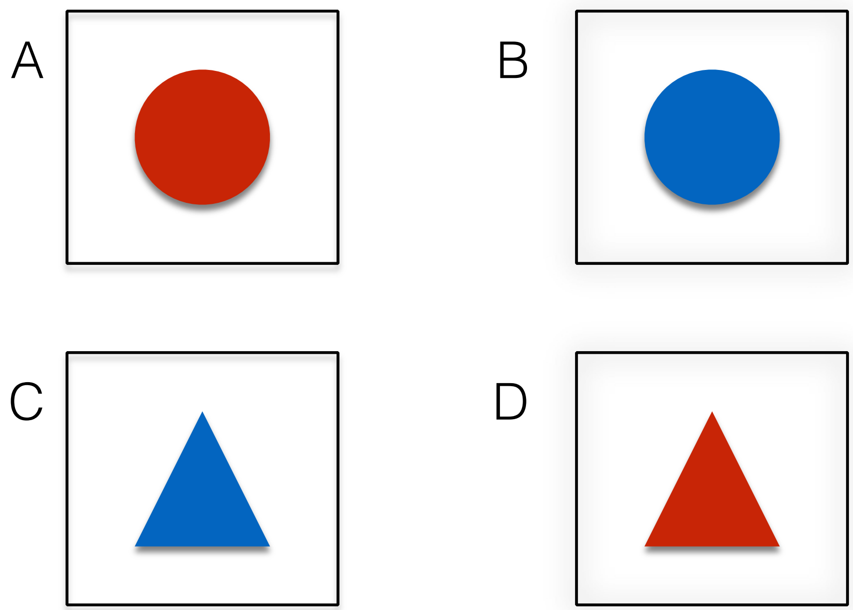

Partition these into two groups…

Courtesy Hamed Pirsiavash

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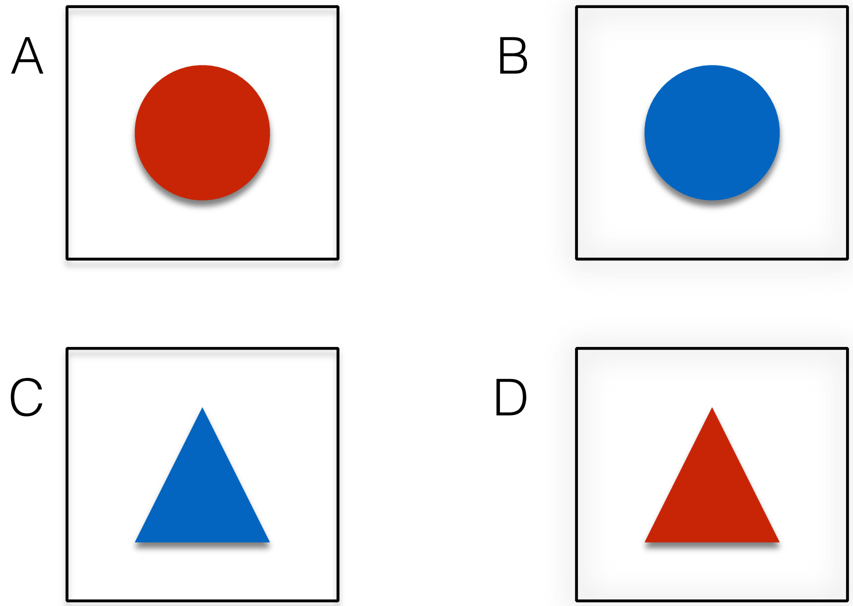

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Who selected red vs. blue?

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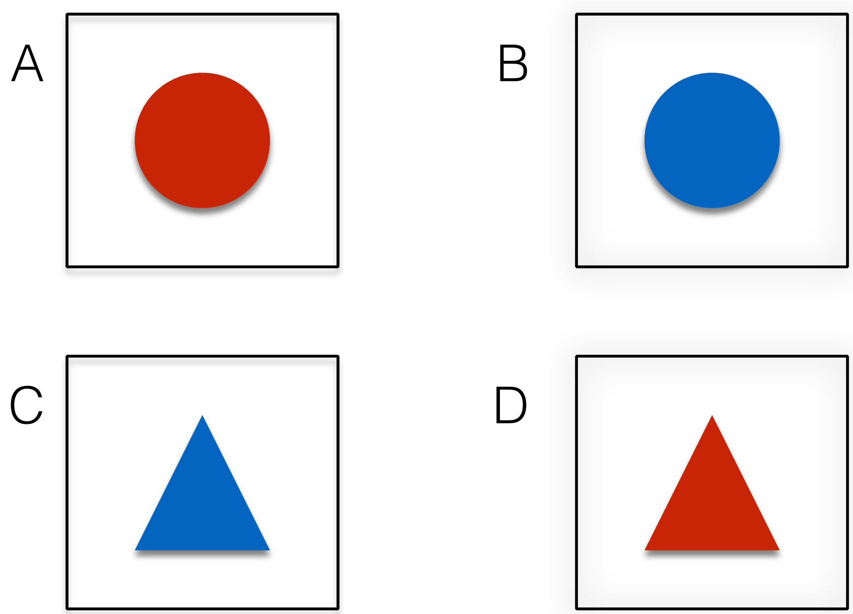

Partition these into two groups

Who selected red vs. blue?

Who selected ● vs. ▲ ?

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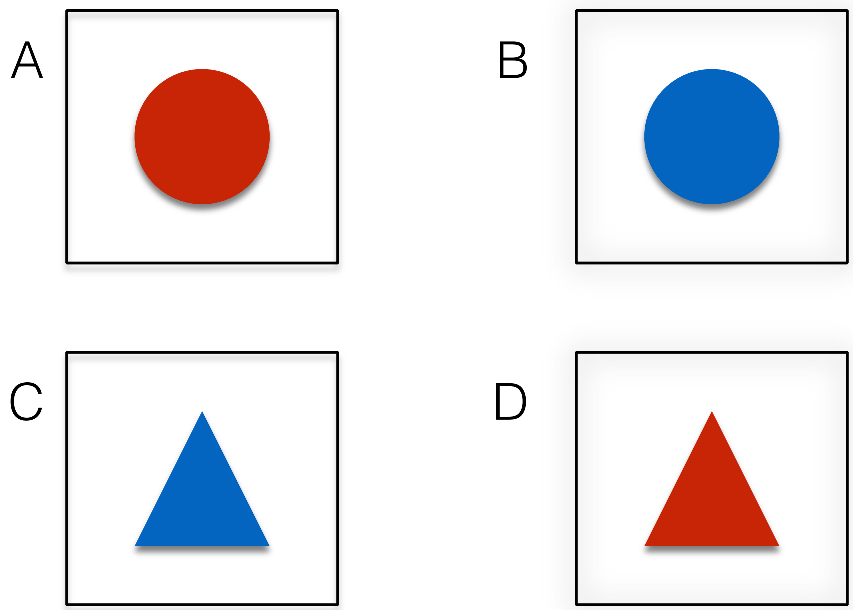

Partition these into two groups

Who selected red vs. blue?

Who selected ● vs. ▲ ?

Tip: Remember how your own biases/interpretation are influencing your approach

Courtesy Hamed Pirsiavash

Today’s Goals:

1. Remember Probability/Statistics
2. Understand Optimizing Empirical Risk

Outline

Review+Extension

Probability

Decision Theory

Loss Functions

Probability Prerequisites

Basic probability axioms and definitions

Joint probability

Probabilistic Independence

Marginal probability

Definition of conditional probability

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Common distributions

Expected Value (of a function) of a Random Variable

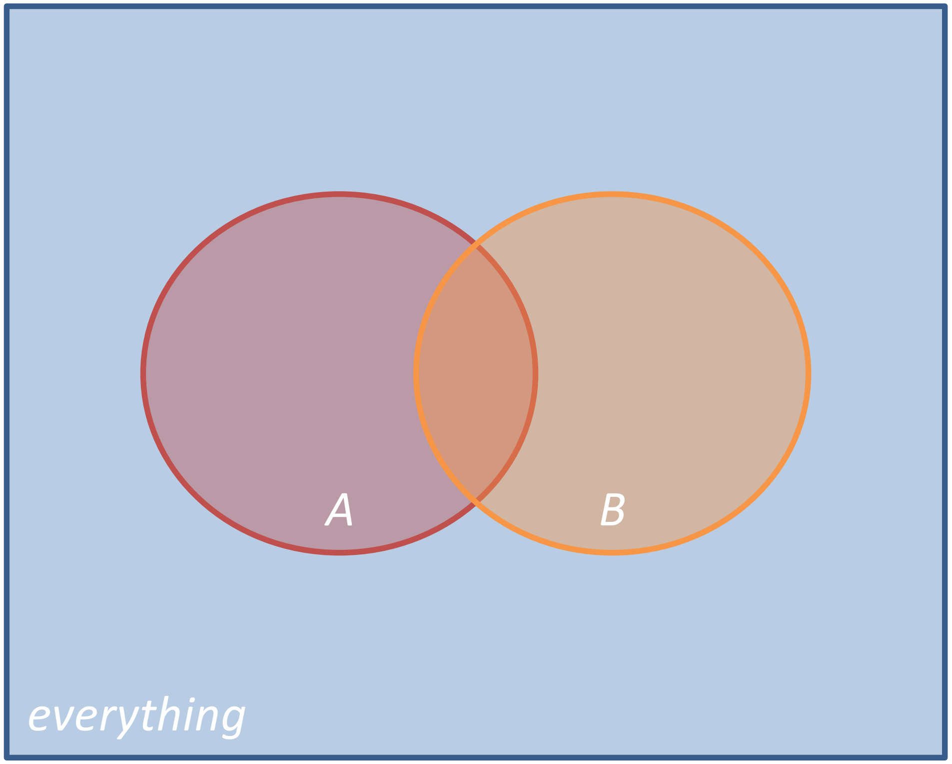
(Most) Probability Axioms

$p(\text{everything}) = 1$

$p(\phi) = 0$

$p(A) \leq p(B)$, when $A \subseteq B$

$p(A \cup B) = p(A) + p(B)$,
when $A \cap B = \phi$



$p(A \cup B) \neq p(A) + p(B)$

$p(A \cup B) = p(A) + p(B) - p(A \cap B)$

Probabilities and Random Variables

Random variables: variables that represent the possible outcomes of some random "process"

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Example #1: A (weighted) coin that can come up heads or tails

- X is a random variable denoting the possible outcomes
- X=HEADS or X=TAILS

Probabilities and Random Variables

Random variables: variables that represent the possible outcomes of some random “process”

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Example #2: Measuring the amount of snow that fell in the last storm

- Y is a random variable denoting the amount snow that fell, in inches
- Y=0, or Y=0.5, or Y=1.0495928591, or Y=10, or …

Probabilities and Random Variables

Random variables: variables that represent the possible outcomes of some random "process"

Example #1: A (weighted) coin that can come up heads or tails

X is a random variable denoting the possible outcomes

X=HEADS or X=TAILS

DISCRETE random variable

Example #2: Measuring the amount of snow that fell in the last storm

Y is a random variable denoting the amount snow that fell, in inches

Y=0, or Y=0.5, or Y=1.0495928591, or Y=10, or …

CONTINUOUS random variable

Random Variables

	If X is a…	
	Discrete random variable	Continuous random variable
The values k that X can take are	Discrete: finite or countably infinite (e.g., integers)	Continuous: uncountably infinite (e.g., real values)

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The function that gives the relative likelihood of a value p(X=k) is a	probability mass function (PMF)	probability density function (PDF)

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The values that PMF/PDF can take are	$0 \leq p(X=k) \leq 1$	$p(X=k) \geq 0$

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We “add” with	Sums ($\sum$)	Integrals ($\int$)

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The values that PMF/PDF can take are	$0 \leq p(X=k) \leq 1$	$p(X=k) \geq 0$
We “add” with	Sums ($\sum$)	Integrals ($\int$)
Our PMF/PDF satisfies p(everything)=1 by	$\sum_k p(X=k) = 1$	$\int p(x)dx = 1$

Probability Prerequisites

Basic probability axioms and definitions

Joint probability

Probabilistic Independence

Marginal probability

Definition of conditional probability

Bayes rule

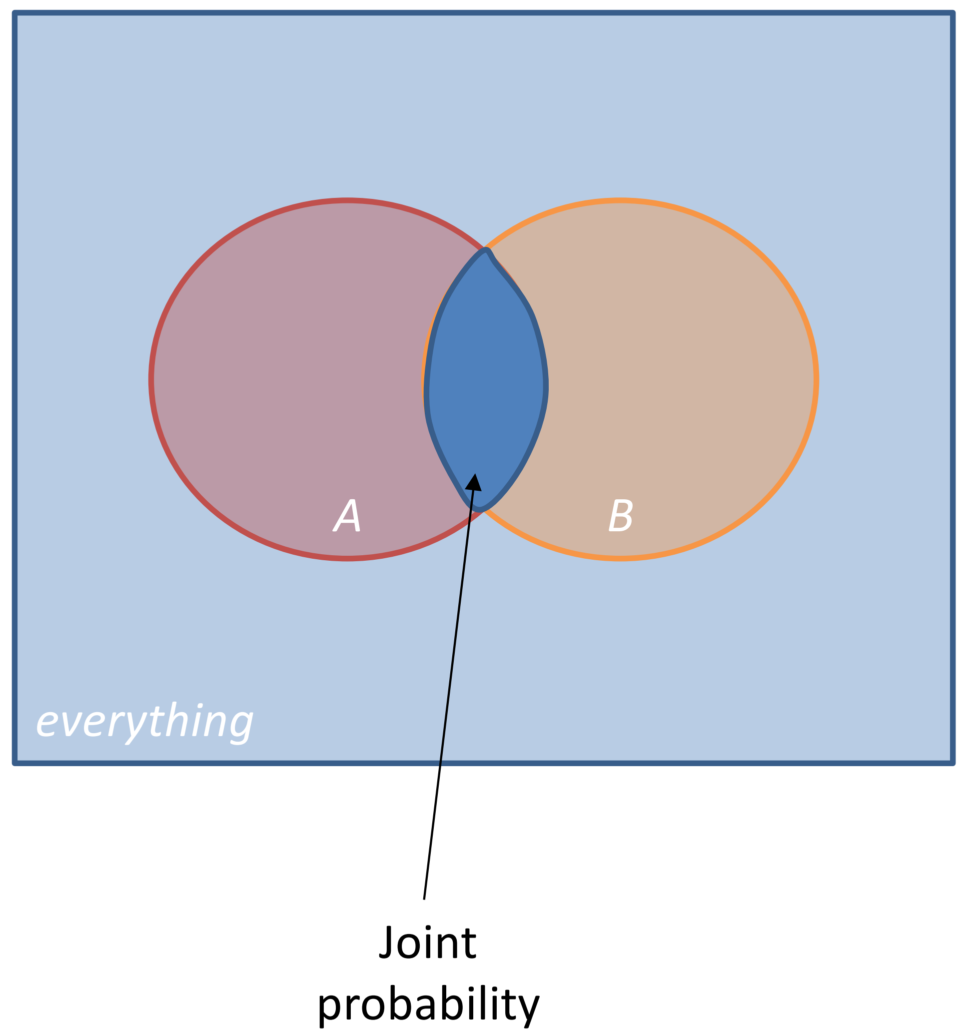
Probability chain rule

Common distributions

Expected Value (of a function) of a Random Variable

Joint Probability

Probability that multiple things "happen together"

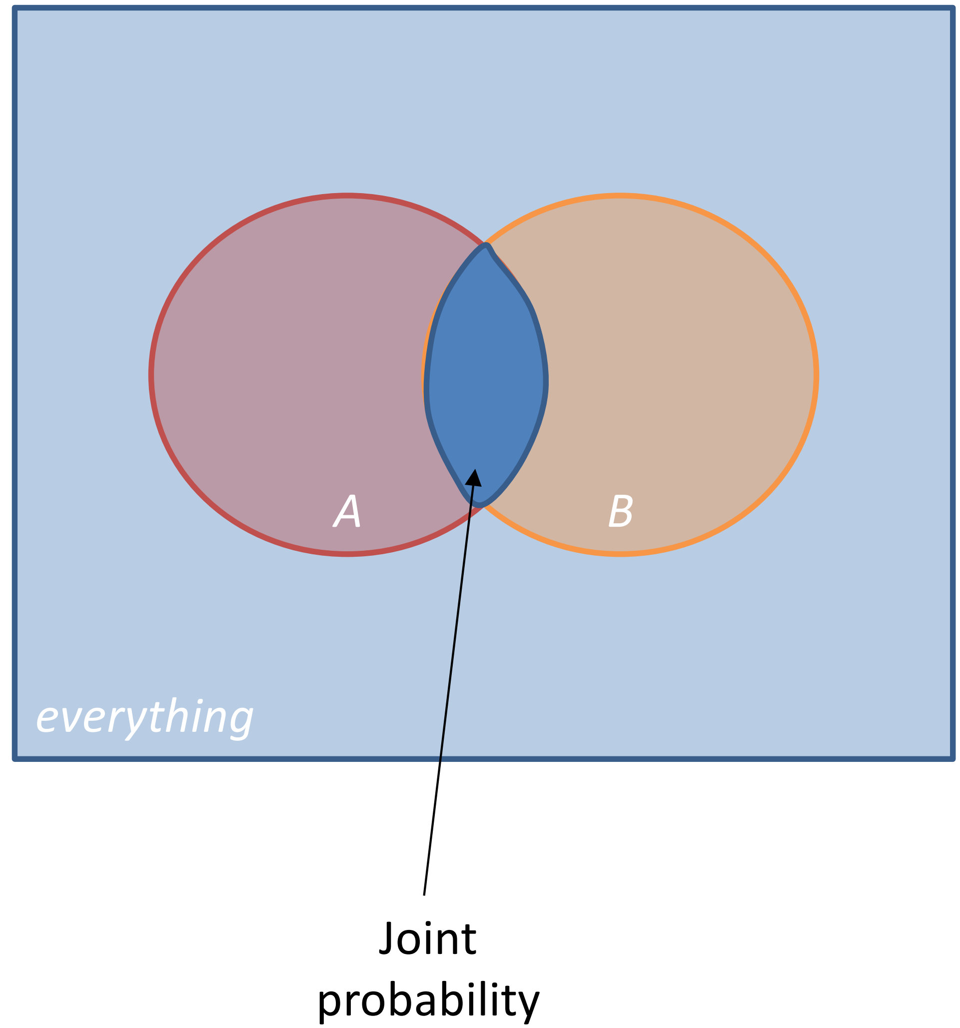


Joint Probability

Probability that multiple things "happen together"

$p(x,y)$, $p(x,y,z)$, $p(x,y,w,z)$

Symmetric: $p(x,y) = p(y,x)$



Joint Probability

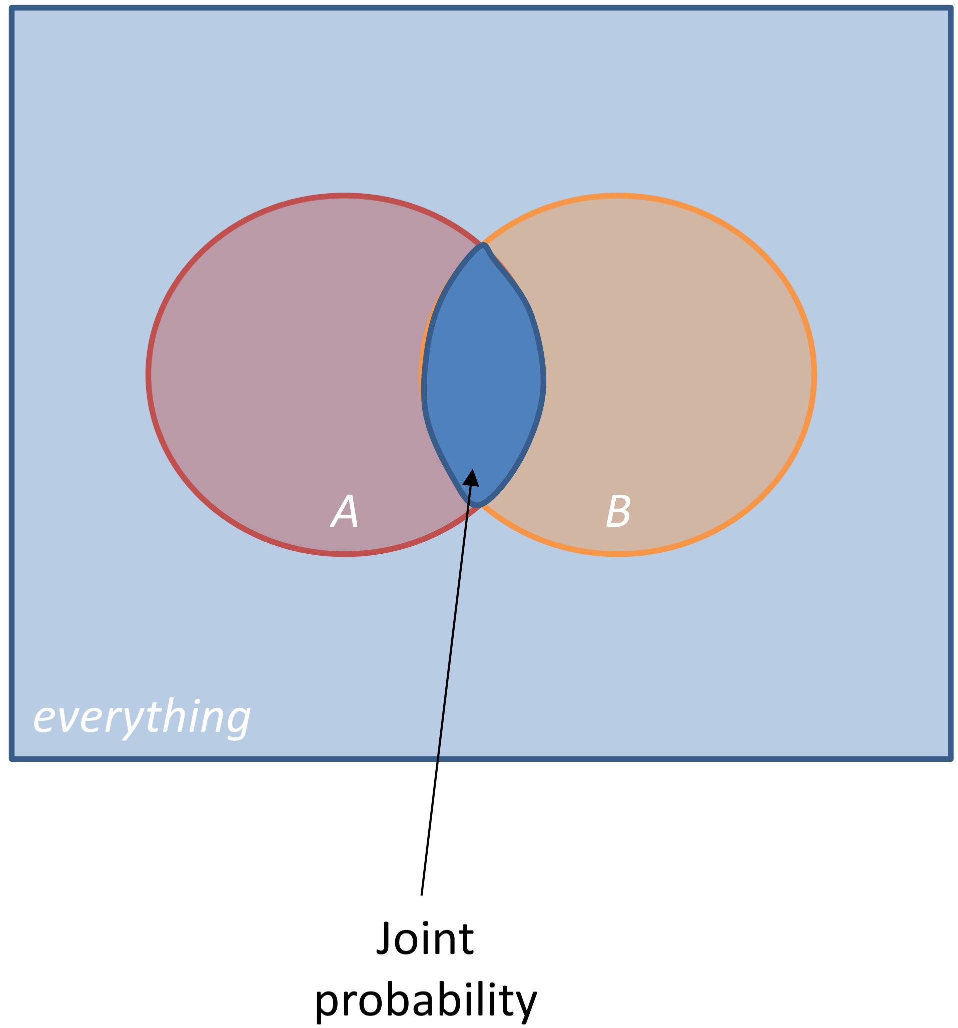
Probability that multiple things "happen together"

p(x,y), p(x,y,z), p(x,y,w,z)

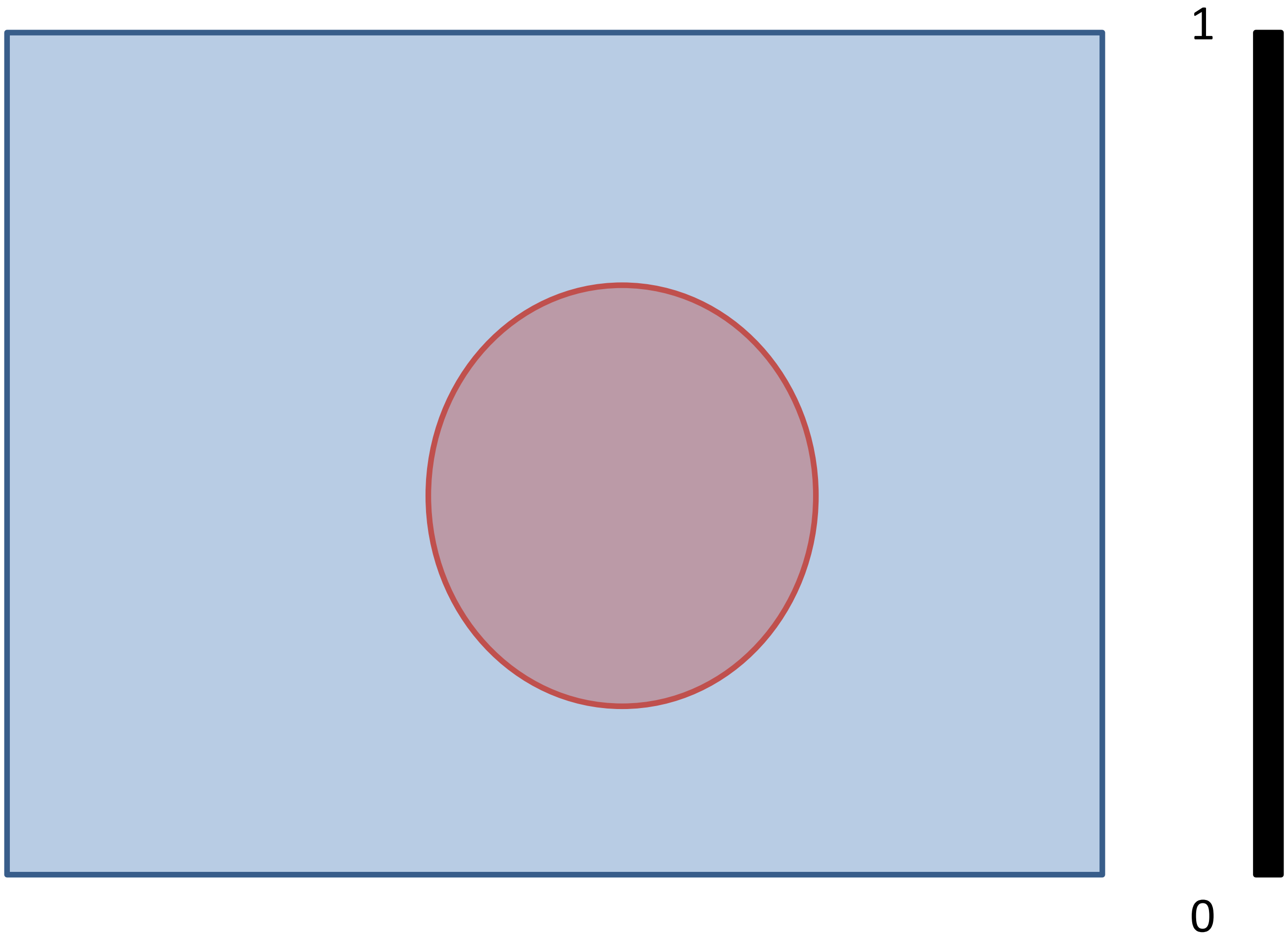
Symmetric: p(x,y) = p(y,x)

Form a table based of outcomes: sum across cells = 1

p(x,y)	Y=0	Y=1
X="cat"	.04	.32
X="dog"	.2	.04
X="bird"	.1	.1
X="human"	.1	.1



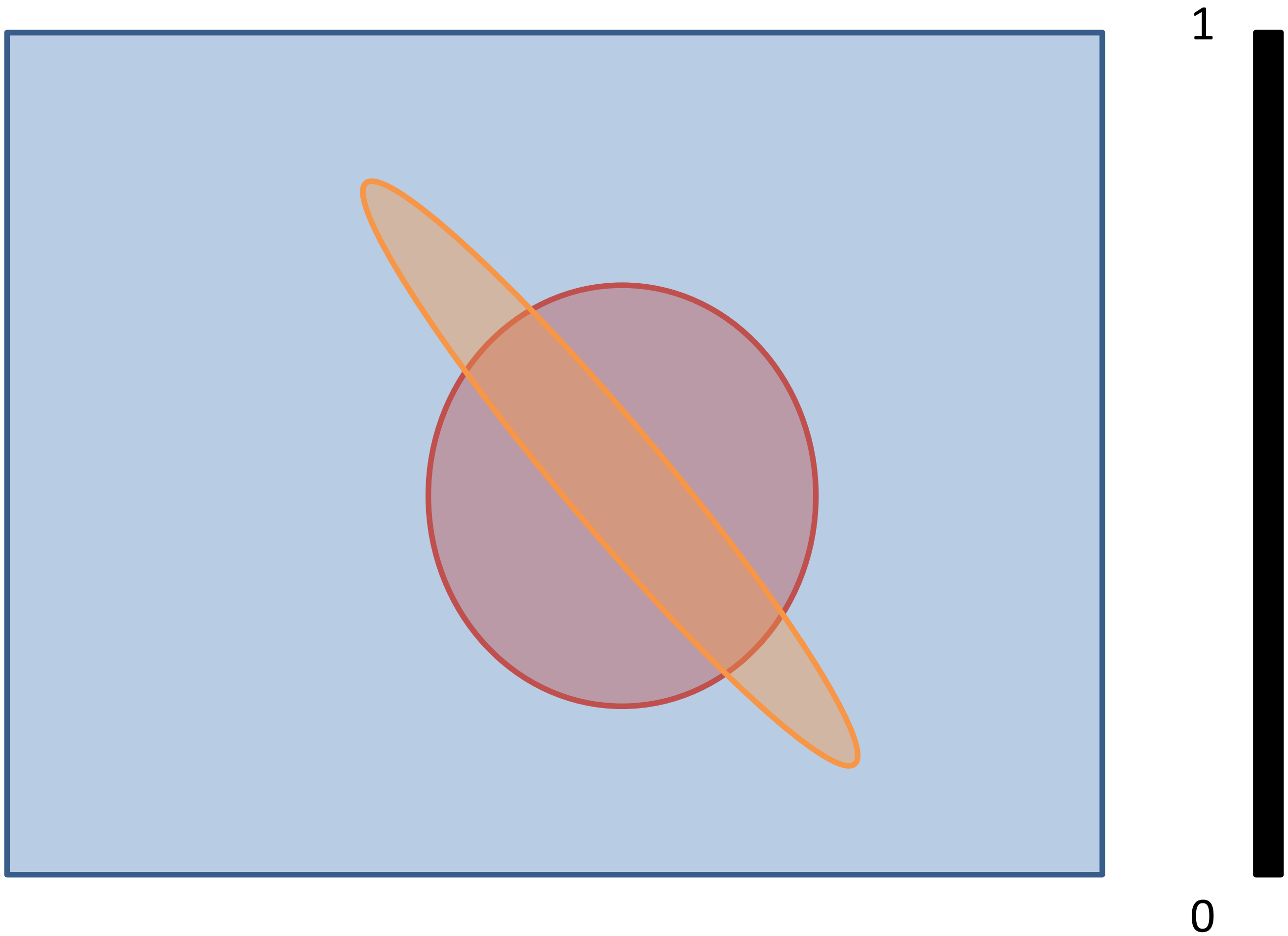
Joint Probabilities



p(A)

what happens as we add conjuncts?

Joint Probabilities



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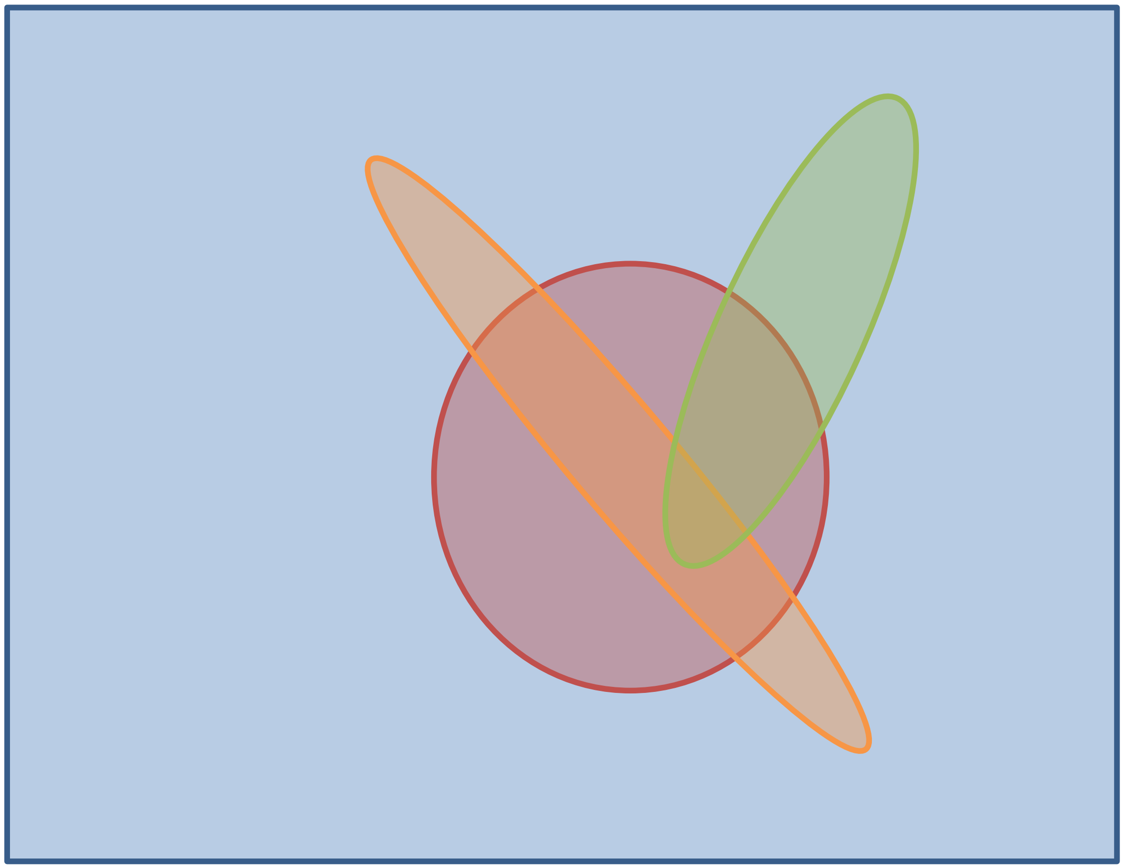


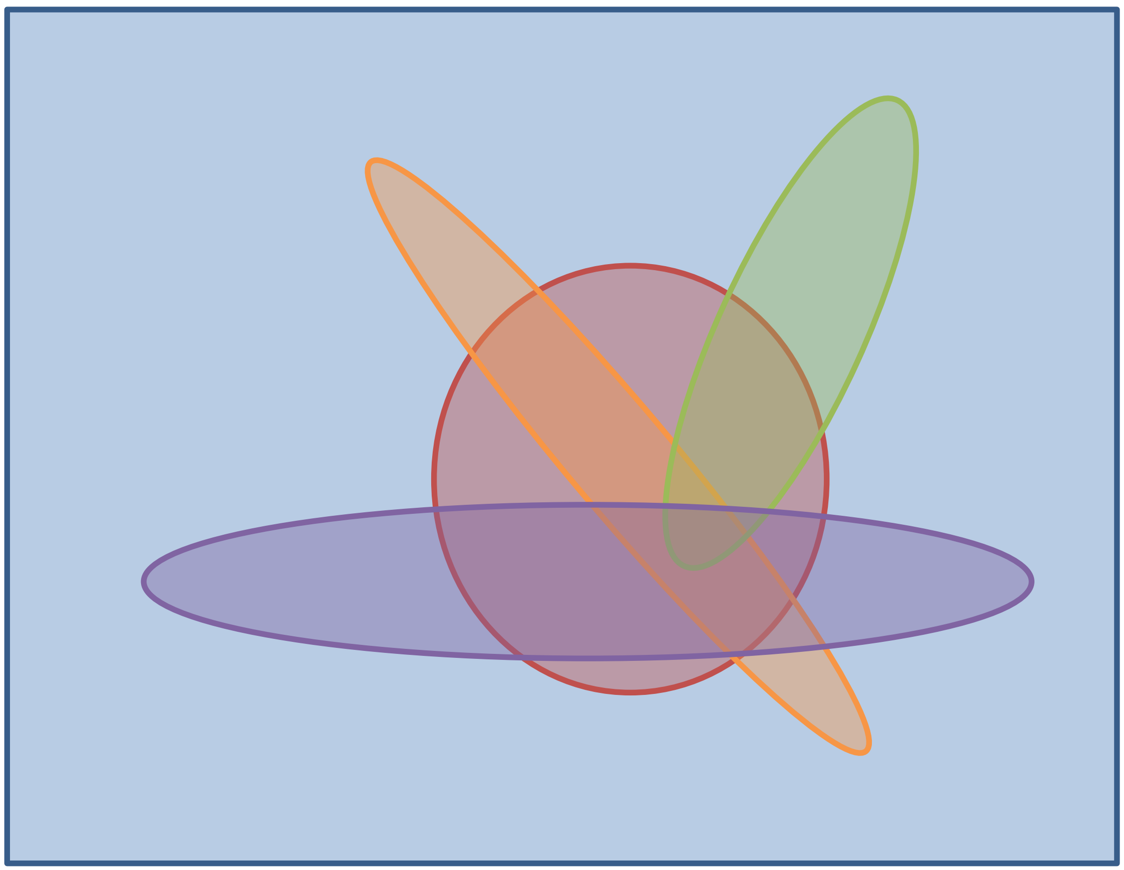
$p(A)$

$p(A, B)$

$p(A, B, C)$

what happens as we add conjuncts?

Joint Probabilities



1

0

p(A)

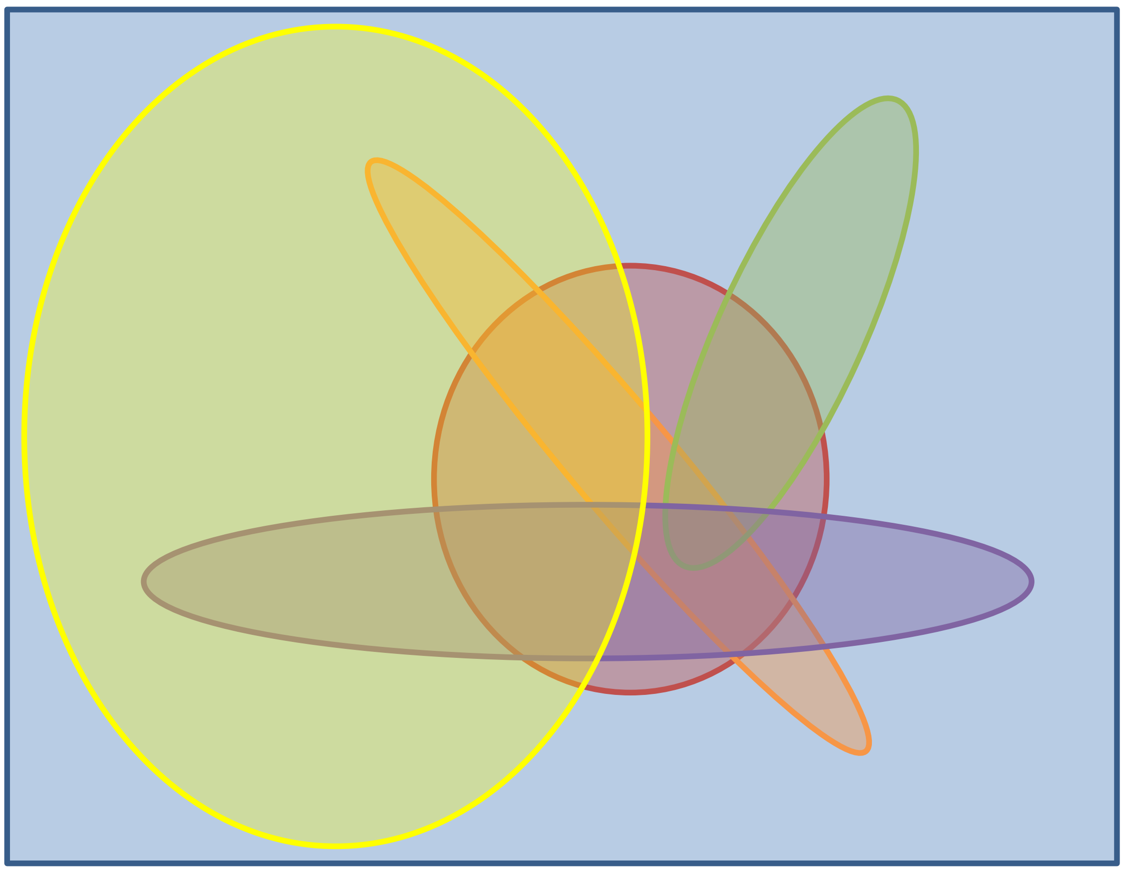
p(A, B)

p(A, B, C)

p(A, B, C, D)

what happens as we add conjuncts?

Joint Probabilities



1

p(A)

p(A, B)

p(A, B, C)

p(A, B, C, D)

p(A, B, C, D, E)

0

what happens as we add conjuncts?

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Expected Value (of a function) of a Random Variable

Probabilistic Independence

Independence: when events can occur and not impact the probability of other events

Formally: $p(x,y) = p(x)*p(y)$

Generalizable to > 2 random variables

Q: Are the results of flipping the same coin twice in succession independent?

Probabilistic Independence

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Formally: $p(x,y) = p(x)*p(y)$

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Q: Are the results of flipping the same coin twice in succession independent?

A: Yes (assuming no weird effects)

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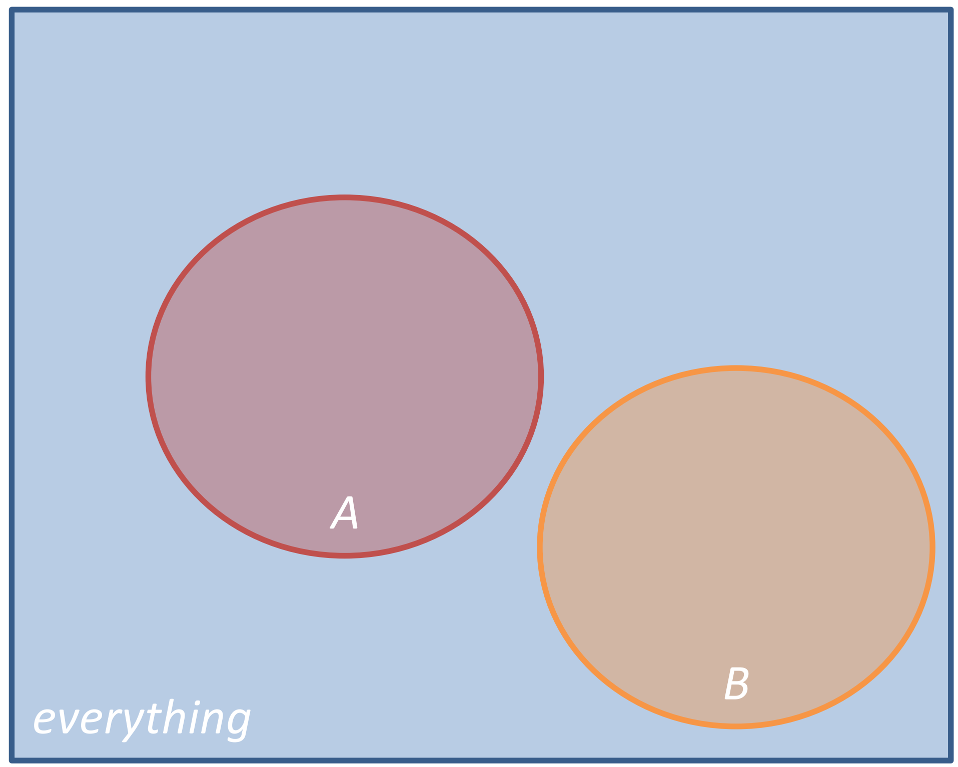

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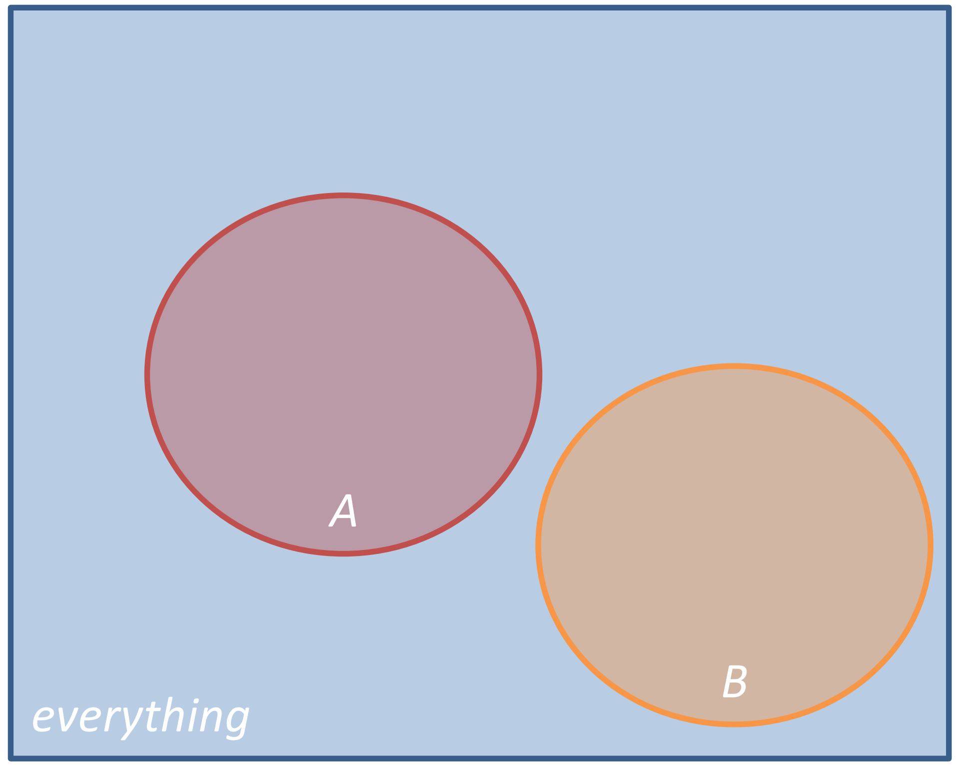

A: No (work it out from p(A,B)) and the axioms

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Q: Are X and Y independent?

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A: No (find the marginal probabilities of p(x) and p(y))

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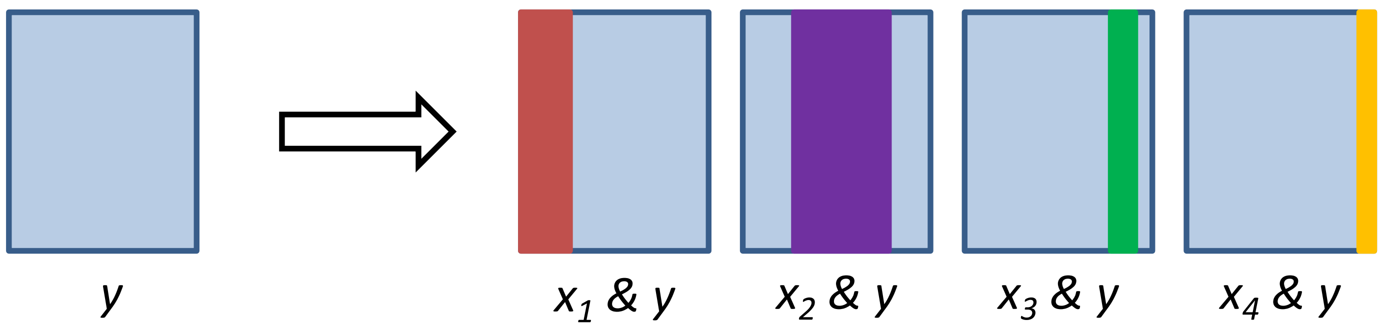
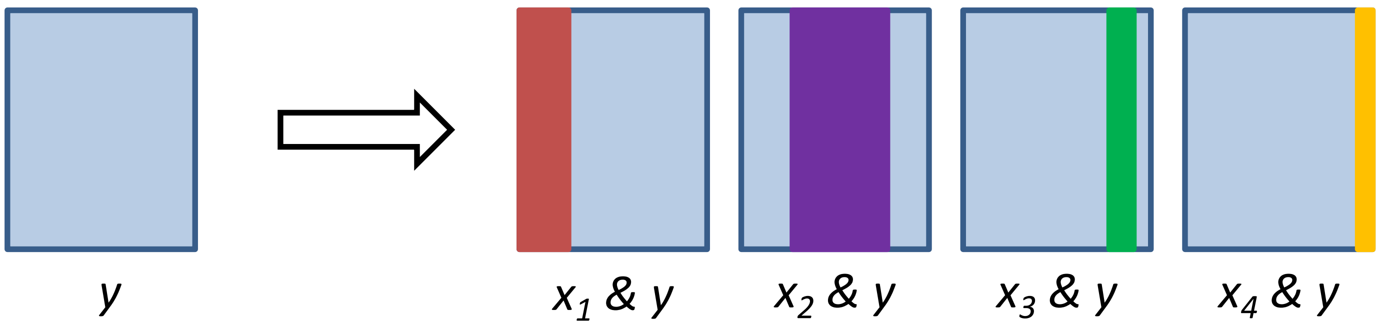
Marginal(ized) Probability: The Discrete Case



Consider the **mutually exclusive** ways that different values of x could occur with y

Q: How do write this in terms of joint probabilities?

Marginal(ized) Probability: The Discrete Case



Consider the **mutually exclusive** ways that different values of x could occur with y

$$p(y) = \sum_{x} p(x, y)$$

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Conditional Probability

$$p(X \mid Y) = \frac{p(X, Y)}{p(Y)}$$

Conditional Probabilities
are Probabilities

Conditional Probability

$$p(X \mid Y) = \frac{p(X, Y)}{p(Y)}$$

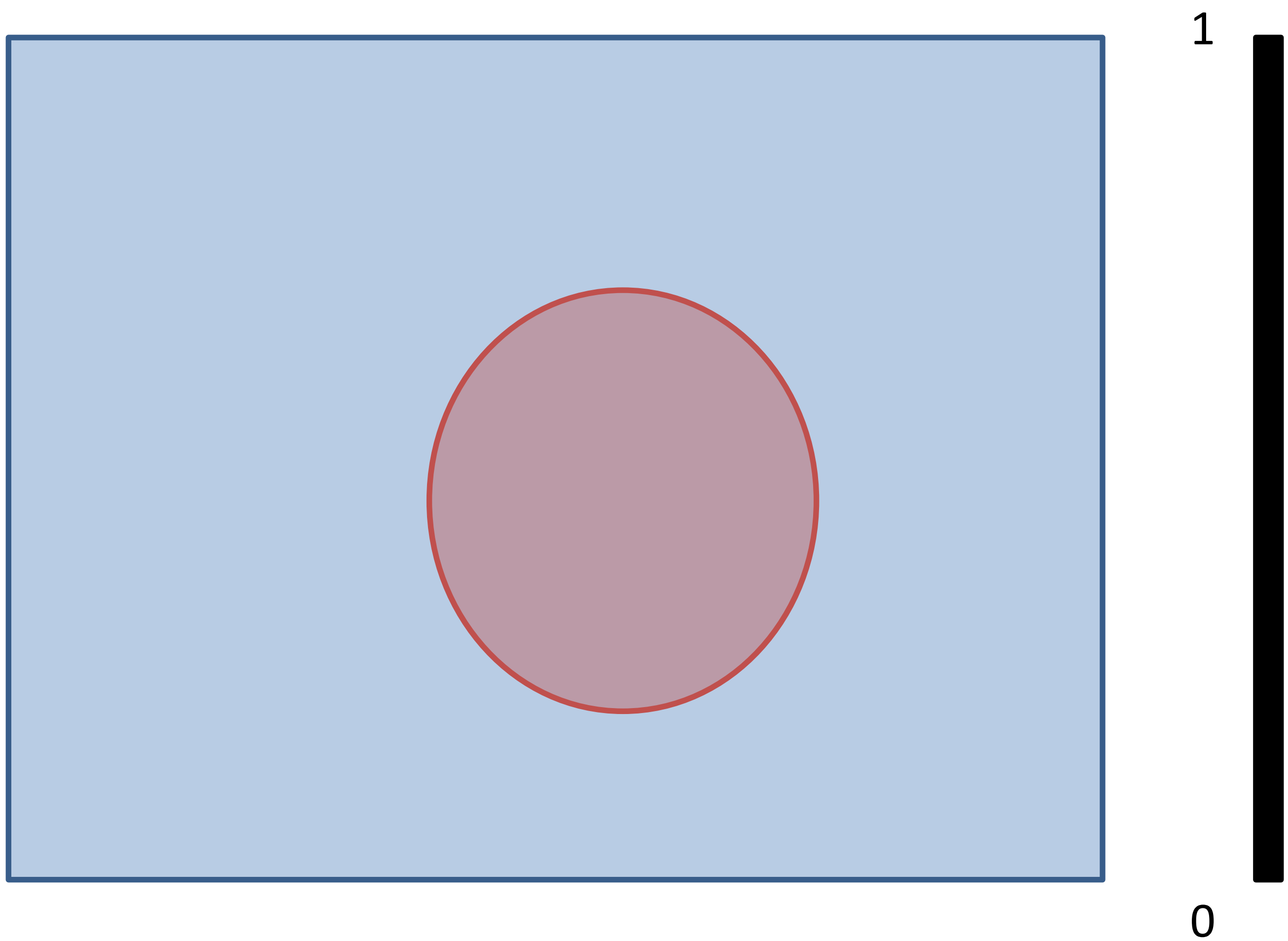
$p(Y) =$ marginal probability of Y

Conditional Probability

$$p(X \mid Y) = \frac{p(X, Y)}{p(Y)}$$

$$p(Y) = \int p(X, Y) dX$$

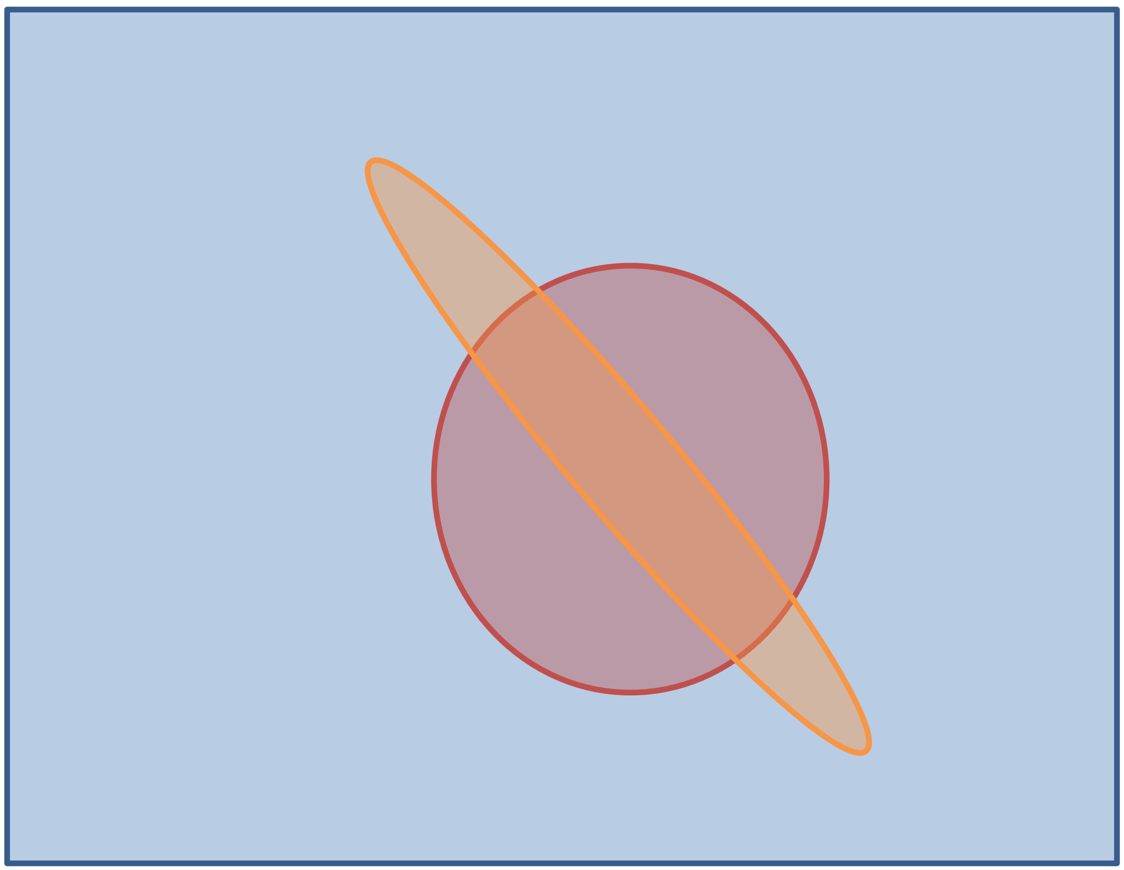
Conditional Probabilities: Changing the Right



p(A)

what happens as we add conjuncts to the right?

Conditional Probabilities: Changing the Right



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p(A | B)

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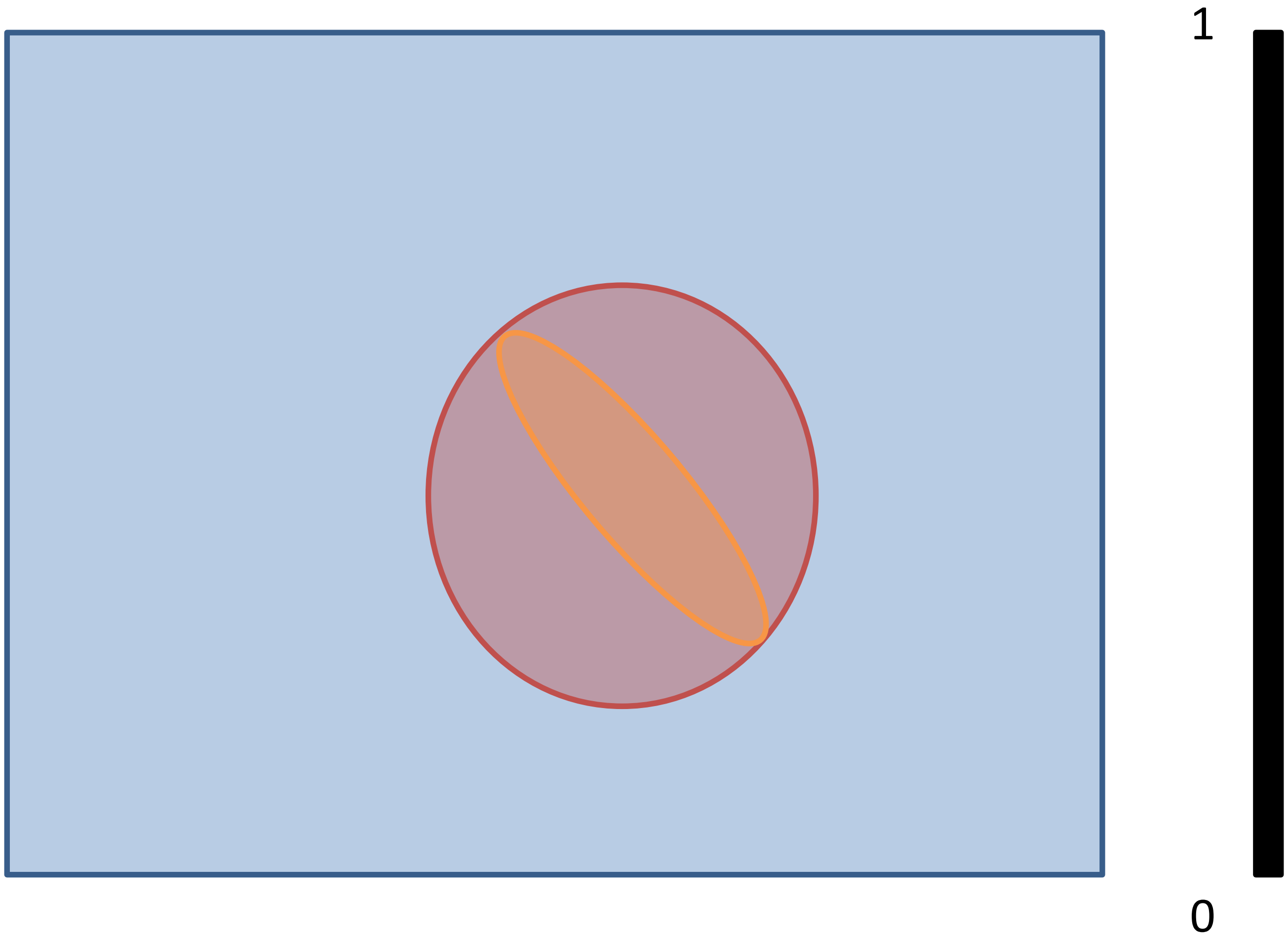
Conditional Probabilities: Changing the Right



$p(A \mid B)$
$p(A)$

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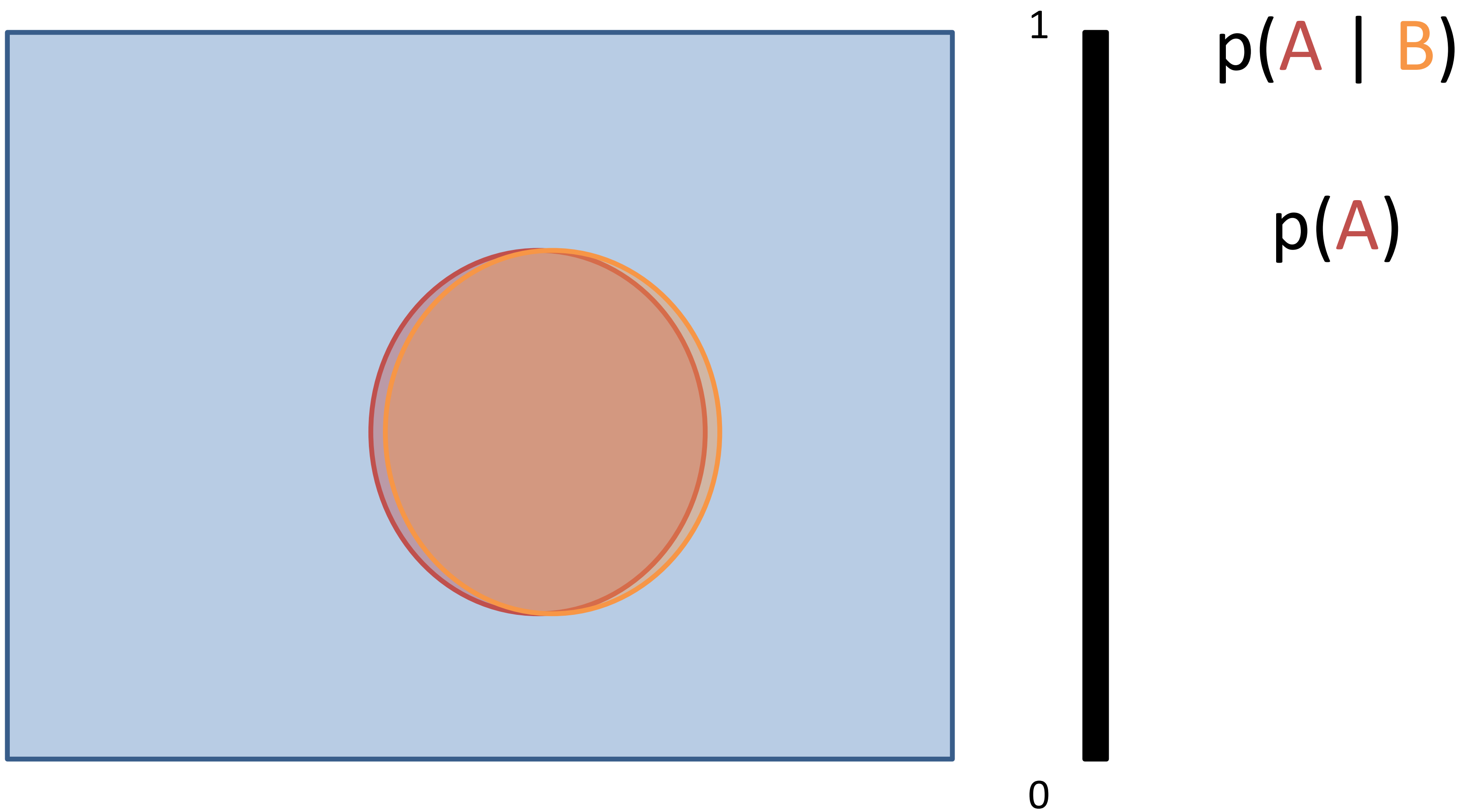
Conditional Probabilities: Changing the Right



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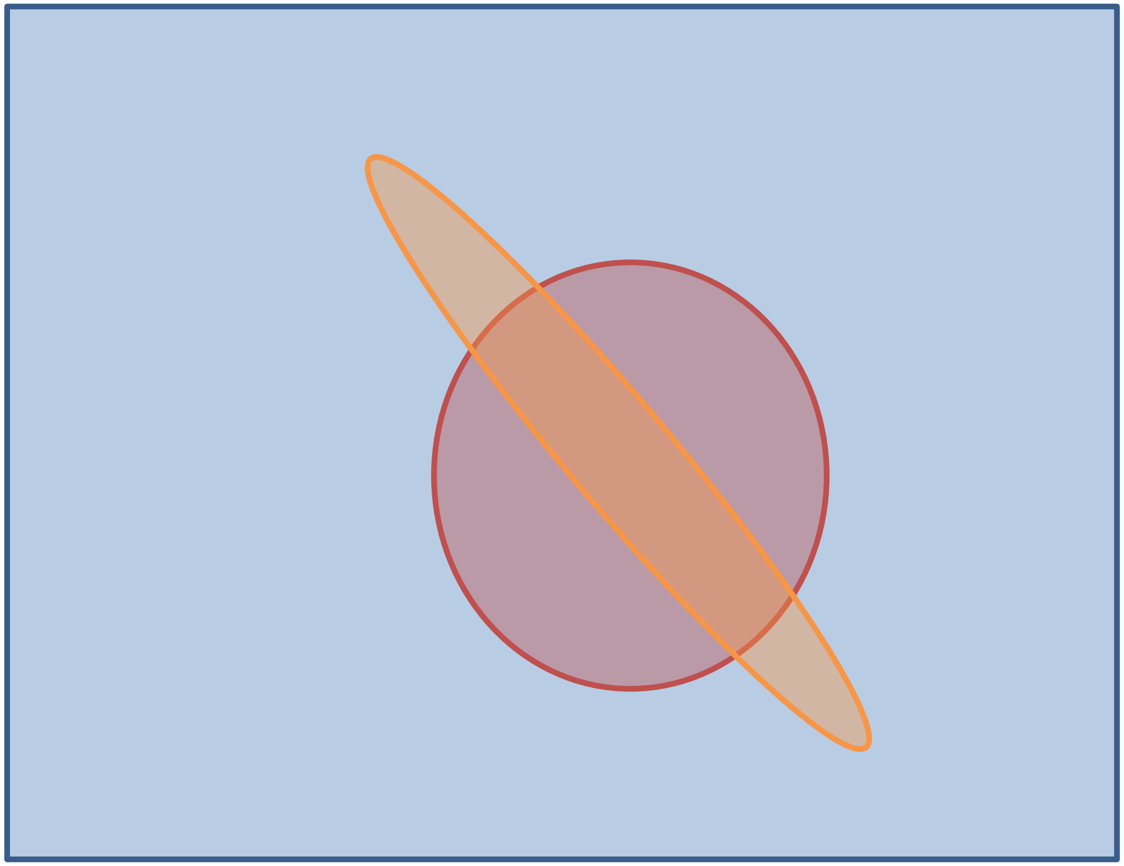
Conditional Probabilities

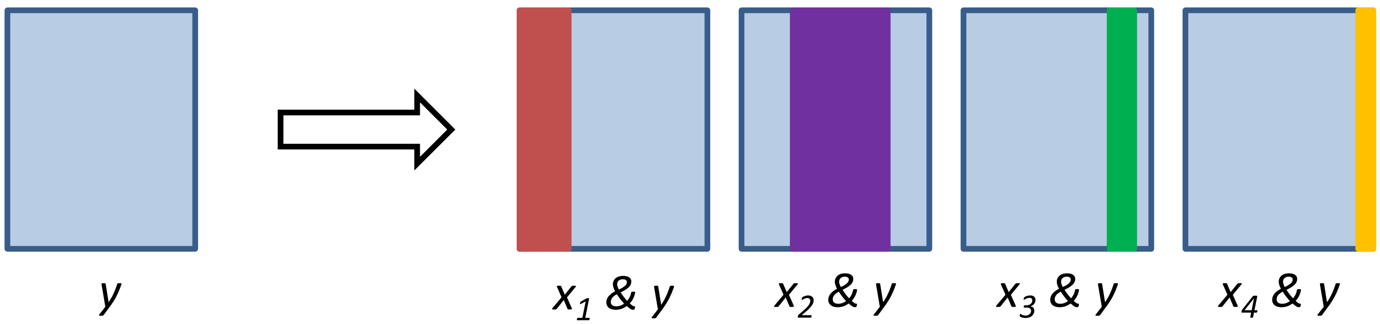


Bias vs. Variance

Lower bias: More specific to what we care about

Higher variance: For fixed observations, estimates become less reliable

Revisiting Marginal Probability: The Discrete Case



$$p(y) = \sum_x p(x, y)$$

$$= \sum_x p(x)p(y \mid x)$$

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Expected Value (of a function) of a Random Variable

Deriving Bayes Rule

Start with conditional
$p(X \mid Y)$

Deriving Bayes Rule

$$p(X \mid Y) = \frac{p(X, Y)}{p(Y)}$$

Solve for p(x,y)

Deriving Bayes Rule

$$p(X \mid Y) = \frac{p(X,Y)}{p(Y)}$$

Solve for p(x,y)

$$p(X,Y) = p(X \mid Y)p(Y)$$

p(x,y) = p(y,x)

$$p(X \mid Y) = \frac{p(Y \mid X) * p(X)}{p(Y)}$$

Bayes Rule

prior probability

likelihood

$$p(X \mid Y) = \frac{p(Y \mid X) * p(X)}{p(Y)}$$

posterior probability

marginal likelihood (probability)

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Probability Chain Rule

$$p(x_1, x_2, \ldots, x_S) =$$
$$p(x_1)p(x_2 \mid x_1)p(x_3 \mid x_1, x_2) \cdots p(x_S \mid x_1, \ldots, x_i) =$$
$$\prod_{i}^{S} p(x_i \mid x_1, \ldots, x_{i-1})$$

extension of Bayes rule

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Distribution Notation

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i.i.d. If $X_1, X_2, \ldots, X_{\mathrm{N}}$ are all independently sampled from $G(\rho)$, they are **i**ndependently and **i**dentically **d**istributed

Common Distributions

Bernoulli/Binomial

Categorical/Multinomial

Poisson

Normal

(Gamma)

Bernoulli: A single draw

- Binary R.V.: 0 (failure) or 1 (success)
- $X \sim \text{Bernoulli}(\rho)$
- $p(X = 1) = \rho, p(X = 0) = 1 - \rho$
- Generally, $p(X = k) = \rho^k (1 - p)^{1-k}$

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Binomial: Sum of N iid Bernoulli draws

- Values X can take: 0, 1, …, N
- Represents number of successes
- $X \sim \text{Binomial}(N, \rho)$
- $p(X = k) = \binom{N}{k} \rho^k (1 - \rho)^{N-k}$

Common Distributions

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Categorical: A single draw

- Finite R.V. taking one of K values: 1, 2, …, K
- $X \sim \text{Cat}(\rho), \rho \in \mathbb{R}^K$
- $p(X = 1) = \rho_1, p(X = 2) = \rho_2, \dots p(X = K) = \rho_K$
- Generally, $p(X = k) = \prod_j \rho_j^{\mathbf{1}[k=j]}$
- $1[c] = \begin{cases} 1, & c \text{ is true} \\ 0, & c \text{ is false} \end{cases}$

Multinomial: Sum of N iid Categorical draws

- Vector of size K representing how often value k was drawn
- $X \sim \text{Multinomial}(N, \rho), \rho \in \mathbb{R}^K$

Common Distributions

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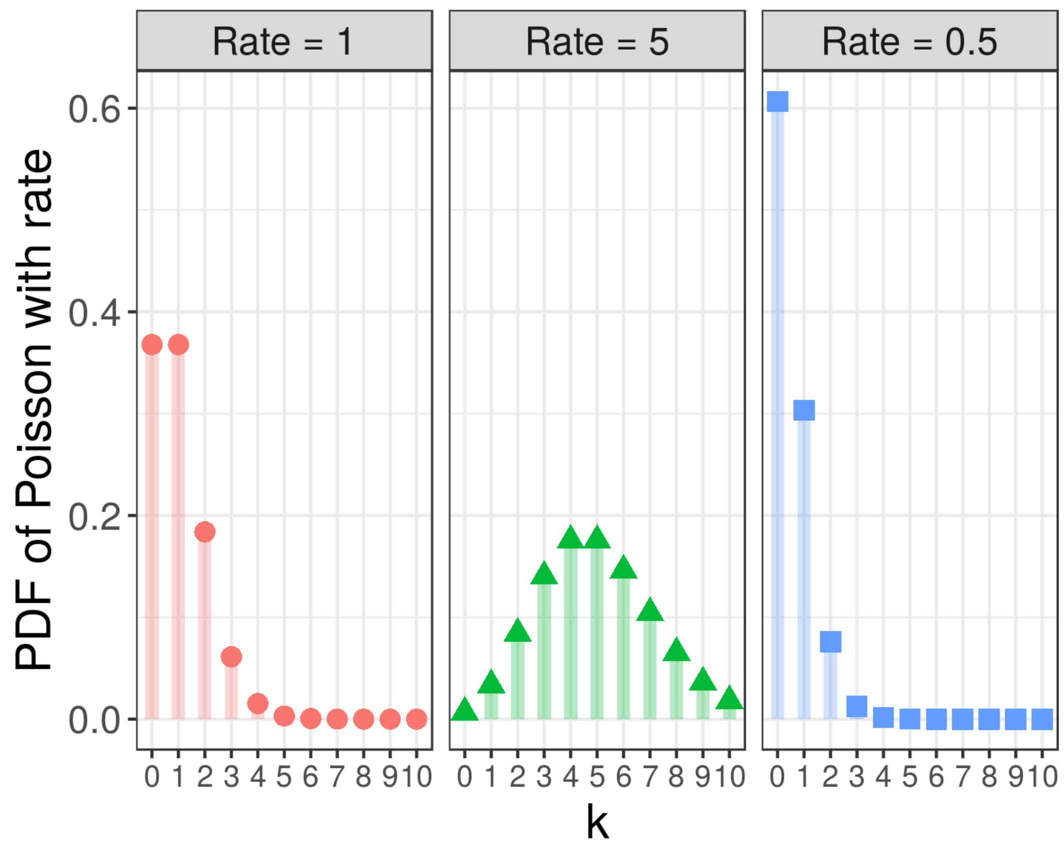
(Gamma)

Poisson

- Finite R.V. taking any integer that is >= 0
- $X \sim \text{Poisson}(\lambda), \lambda \in \mathbb{R}$ is the "rate"
- $p(X = k) = \dfrac{\lambda^k \exp(-\lambda)}{k!}$



Common Distributions

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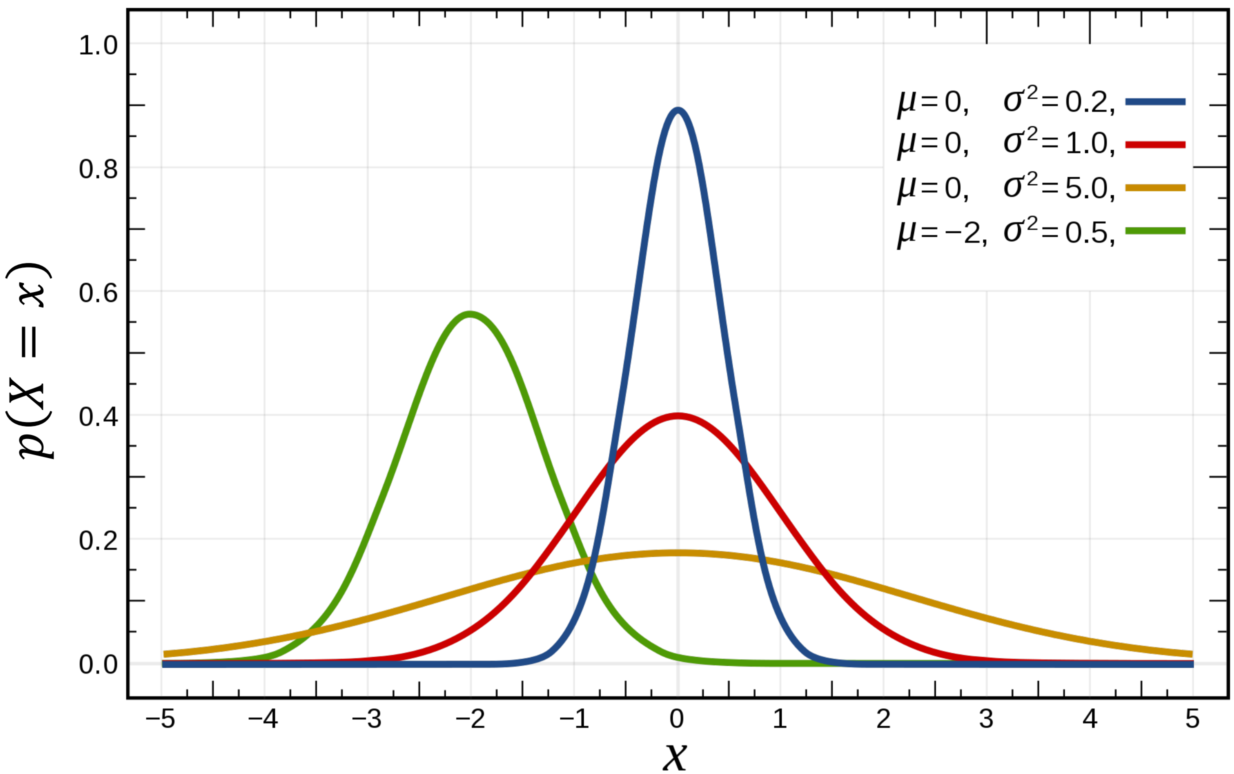
Normal

(Gamma)

Normal

- Real R.V. taking any real number
- $X \sim \text{Normal}(\mu, \sigma)$, μ is the mean, σ is the standard deviation
- $p(X = x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)$



https://upload.wikimedia.org/wikipedia/commons/thumb/7/74/Normal_Distribution_PDF.svg/1920px-Normal_Distribution_PDF.svg.png

Probability Prerequisites

Basic probability axioms and definitions

Joint probability

Probabilistic Independence

Marginal probability

Definition of conditional probability

Bayes rule

Probability chain rule

Common distributions

Expected Value (of a function) of a Random Variable

Expected Value of a Random Variable

random variable

$$X \sim p(\cdot)$$

Expected Value of a Random Variable

random variable

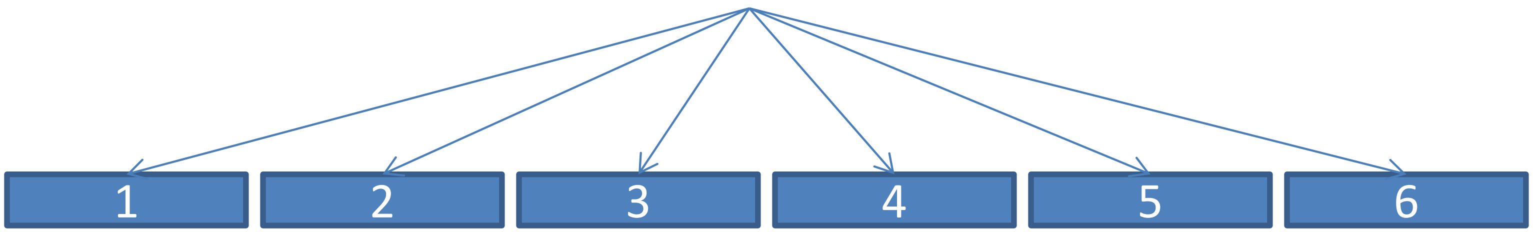
$$X \sim p(\cdot)$$

$$\mathbb{E}[X] = \sum_{x} x \, p(x)$$

expected value (distribution p is implicit)

Expected Value: Example

uniform distribution of number of cats I have

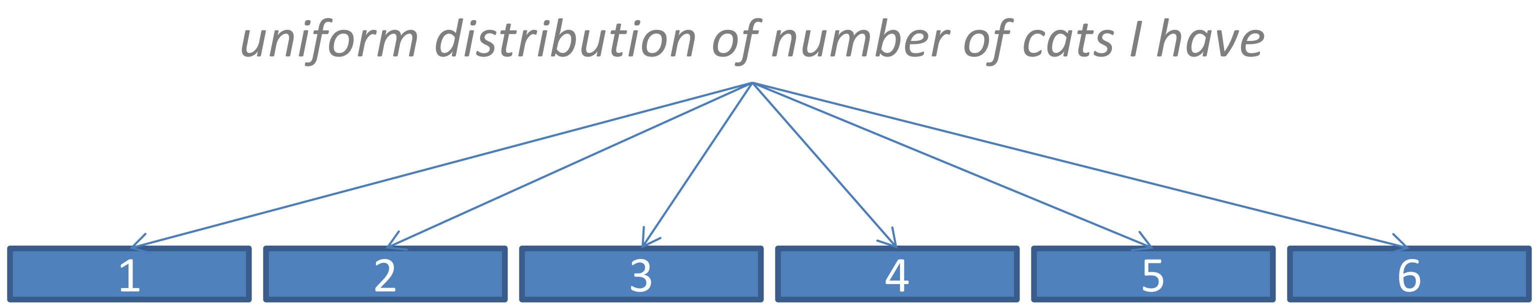


$$\mathbb{E}[X] = \sum_x x \, p(x)$$

1/6 * 1 +
1/6 * 2 +
1/6 * 3 +
1/6 * 4 +
1/6 * 5 +
1/6 * 6

= 3.5

Expected Value: Example



$$\mathbb{E}[X] = \sum_{x} x\, p(x)$$

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1/6 * 2 +
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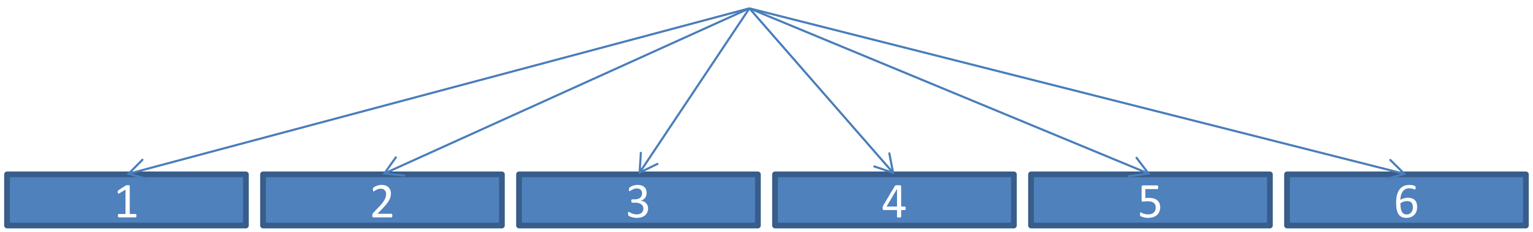
1/6 * 4 +
1/6 * 5 +
1/6 * 6

= 3.5

Q: What common distribution is this?

Expected Value: Example

uniform distribution of number of cats I have



$$\mathbb{E}[X] = \sum_x x \, p(x)$$

1/6 * 1 +
1/6 * 2 +
1/6 * 3 +
1/6 * 4 +
1/6 * 5 +
1/6 * 6

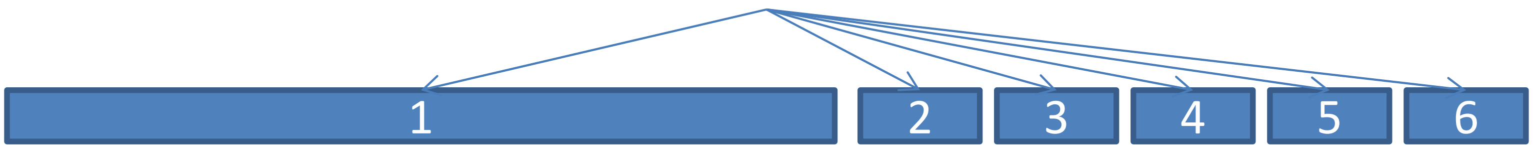
= 3.5

Q: What common distribution is this?

A: Categorical

Expected Value: Example 2

non-uniform distribution of number of cats a normal cat person has



$$\mathbb{E}[X] = \sum_x x\, p(x)$$

1/2 * 1 +
1/10 * 2 +
1/10 * 3 +
1/10 * 4 +
1/10 * 5 +
1/10 * 6

= 2.5

Expected Value of a Function of a Random Variable

$$X \sim p(\cdot)$$

$$\mathbb{E}[X] = \sum_x x\, p(x)$$

$$\mathbb{E}[f(X)] = ???$$

Expected Value of a Function of a Random Variable

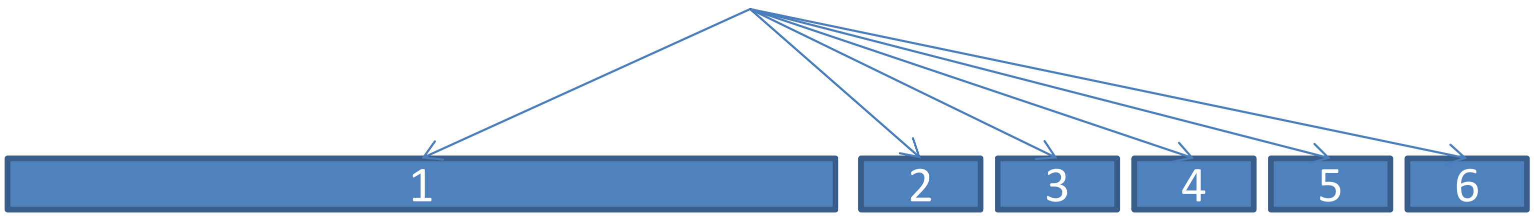
$$X \sim p(\cdot)$$

$$\mathbb{E}[X] = \sum_x x\, p(x)$$

$$\mathbb{E}[f(X)] = \sum_x f(x)\, p(x)$$

Expected Value of Function: Example

non-uniform distribution of number of cats I start with



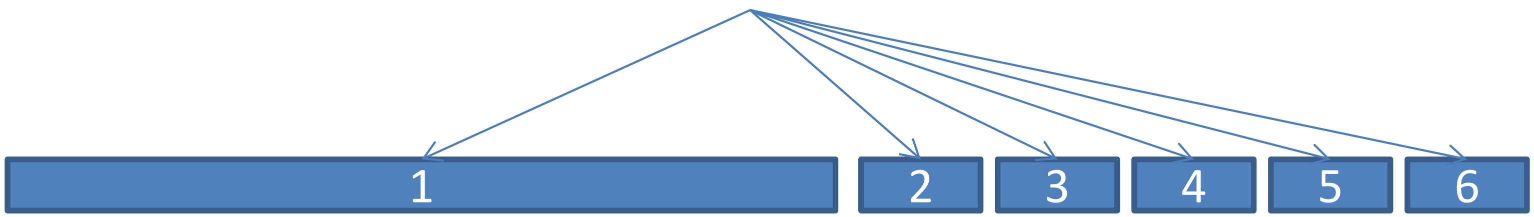
What if each cat magically becomes two?

$$f(k) = 2^k$$

$$\mathbb{E}[f(X)] = \sum_x f(x)\, p(x)$$

Expected Value of Function: Example

non-uniform distribution of number of cats I start with



What if each cat magically becomes two?

$$f(k) = 2^k$$

$$\mathbb{E}[f(X)] = \sum_x f(x)\, p(x) = \sum_x 2^x p(x)$$

$$\begin{array}{c} 1/2 * 2^1 + \\ 1/10 * 2^2 + \\ 1/10 * 2^3 + \\ 1/10 * 2^4 + \\ 1/10 * 2^5 + \\ 1/10 * 2^6 \end{array} \quad = 13.4$$

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Outline

Review+Extension

Probability

Decision Theory

Loss Functions

Decision Theory

"Decision theory is trivial, apart from the computational details" – MacKay, ITILA, Ch 36

Input: **x** ("state of the world")

Output: a decision $\hat{y}$

Decision Theory

"Decision theory is trivial, apart from the computational details" – MacKay, ITILA, Ch 36

Input: $\mathbf{x}$ ("state of the world")

Output: a decision $\hat{y}$

Requirement 1: a decision (hypothesis) function $h(\mathbf{x})$ to produce $\hat{y}$

Decision Theory

"Decision theory is trivial, apart from the computational details" – MacKay, ITILA, Ch 36

Input: **x** ("state of the world")

Output: a decision ŷ

Requirement 1: a decision (hypothesis) function h(**x**) to produce ŷ

Requirement 2: a function ℓ(y, ŷ) telling us how wrong we are

Decision Theory

"Decision theory is trivial, apart from the computational details" – MacKay, ITILA, Ch 36

Input: **x** ("state of the world")

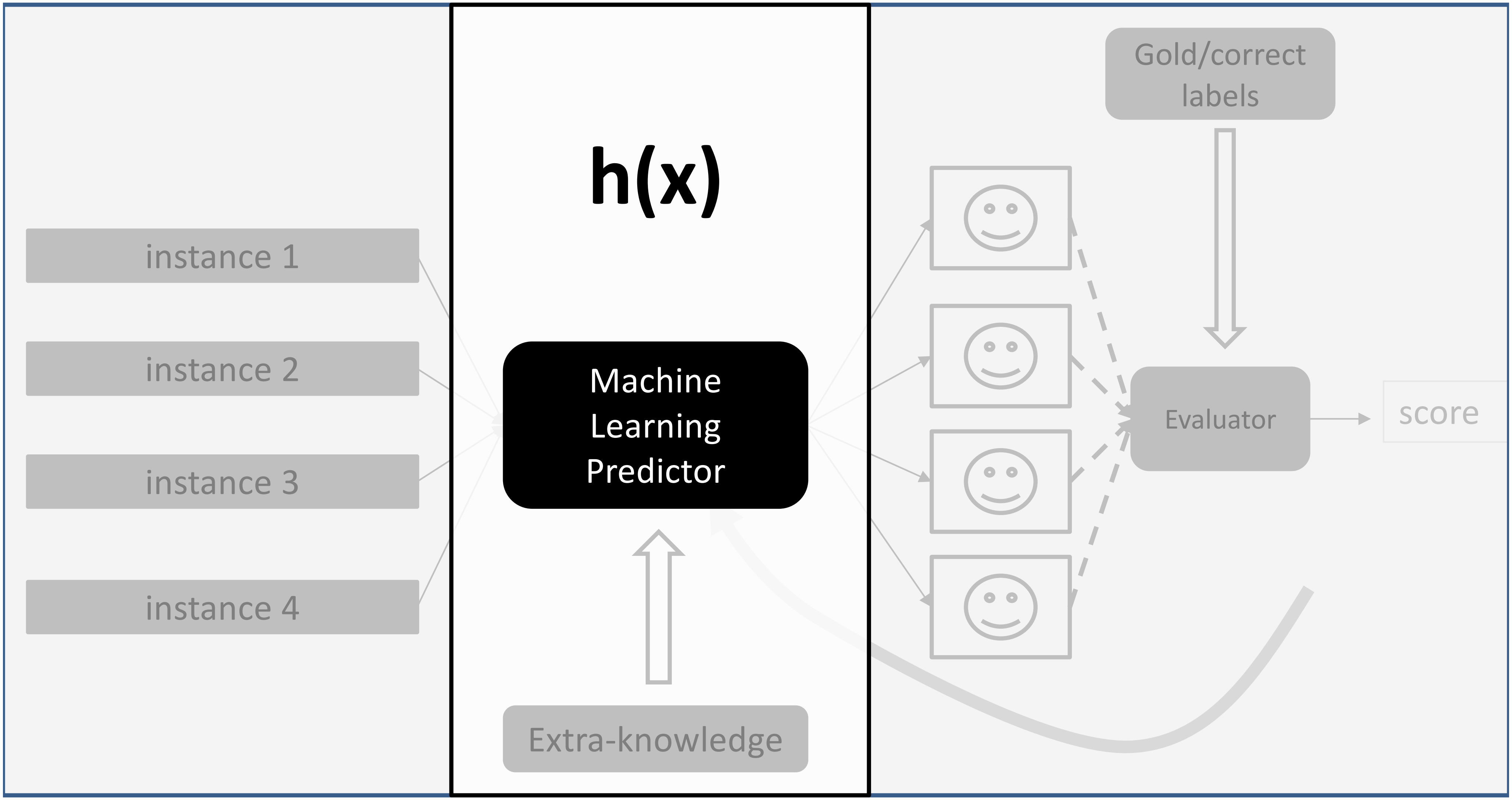
Output: a decision ŷ

Requirement 1: a decision (hypothesis) function h(**x**) to produce ŷ

Requirement 2: a loss function ℓ(y, ŷ) telling us how wrong we are

Goal: minimize our *expected* loss across any possible input

Requirement 1: Decision Function



h(x) is our predictor (classifier, regression model, clustering model, etc.)

Requirement 2: Loss Function

"ell" (fancy l character)

predicted label/result

$$\ell(y, \hat{y}) \geq 0$$

"correct" label/result

optimize ℓ?

- minimize
- ~~maximize~~

loss: A function that tells you how much to penalize a prediction $\hat{y}$ from the correct answer y

Requirement 2: Loss Function

"ell" (fancy l character)

predicted label/result

Negative ℓ ($-\ell$) is called a *utility* or *reward* function

$$\ell(y, \hat{y}) \geq 0$$

"correct" label/result

loss: A function that tells you how much to penalize a prediction $\hat{y}$ from the correct answer y

Decision Theory

minimize expected loss across any possible input

$$\arg\min_{\hat{y}} \mathbb{E}[\ell(y, \hat{y})]$$

Risk Minimization

minimize expected loss across any possible input

$$\arg\min_{\hat{y}} \mathbb{E}[\ell(y, \hat{y})] = \arg\min_{h} \mathbb{E}[\ell(y, h(\boldsymbol{x}))]$$

a *particular*, unspecified input pair (**x**,y)… but we want any possible pair

Decision Theory

minimize expected loss across any possible input

$$\arg\min_{\hat{y}} \mathbb{E}[\ell(y, \hat{y})] =$$

$$\arg\min_{h} \mathbb{E}[\ell(y, h(\boldsymbol{x}))] =$$

$$\operatorname*{argmin}_{\mathrm{h}} \mathbb{E}_{(\boldsymbol{x}, y) \sim P}\left[\ell\left(y, h(\boldsymbol{x})\right)\right]$$

Assumption: there exists *some* true (but likely unknown) distribution *P* over inputs **x** and outputs y

Risk Minimization

minimize expected loss across any possible input

$$\arg\min_{\hat{y}} \mathbb{E}[\ell(y, \hat{y})] =$$

$$\arg\min_{h} \mathbb{E}[\ell(y, h(\boldsymbol{x}))] =$$

$$\underset{\mathrm{h}}{\operatorname{argmin}}\ \mathbb{E}_{(\boldsymbol{x},y)\sim P}\left[\ell\big(y, h(\boldsymbol{x})\big)\right] =$$

$$\underset{\mathrm{h}}{\operatorname{argmin}} \int \ell\big(y, h(\boldsymbol{x})\big) P(\boldsymbol{x}, y)\, d(\boldsymbol{x}, y)$$

Risk Minimization

minimize expected loss across any possible input

$$\arg\min_{\hat{y}} \mathbb{E}[\ell(y, \hat{y})] =$$

$$\arg\min_{h} \mathbb{E}[\ell(y, h(\boldsymbol{x}))] =$$

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$$\operatorname*{argmin}_{\mathrm{h}} \int \ell\left(y, h(\boldsymbol{x})\right) P(\boldsymbol{x}, y)\, d(\boldsymbol{x}, y)$$

we don't know this distribution*!

**we could try to approximate it analytically*

Empirical Risk Minimization

minimize expected loss across our observed input

$$\arg\min_{\hat{y}} \mathbb{E}[\ell(y, \hat{y})] =$$

$$\arg\min_{h} \mathbb{E}[\ell(y, h(\boldsymbol{x}))] =$$

$$\operatorname*{argmin}_{\mathrm{h}} \mathbb{E}_{(\boldsymbol{x}, y) \sim P}\left[\ell\left(y, h(\boldsymbol{x})\right)\right] \approx$$

$$\operatorname*{argmin}_{\mathrm{h}} \frac{1}{N} \sum_{i=1}^{N} \ell\left(y_i, h(\boldsymbol{x}_i)\right)$$

Empirical Risk Minimization

minimize expected loss across our observed input

$$\underset{h}{\operatorname{argmin}} \sum_{i=1}^{N} \ell\left(y_i, h(\boldsymbol{x}_i)\right)$$

our
classifier/predictor

controlled by our
parameters θ

change θ →
change the behavior of the classifier

Best Case: Optimize Empirical Risk with Gradients

$$\operatorname*{argmin}_{\mathrm{h}} \sum_{i=1}^{N} \ell\big(y_i, h_\theta(\boldsymbol{x}_i)\big)$$

change θ →
change the behavior of the classifier

$$\Downarrow$$

$$\operatorname*{argmin}_{\theta} \sum_{i=1}^{N} \ell\big(y_i, h_\theta(\boldsymbol{x}_i)\big)$$

Best Case: Optimize Empirical Risk with Gradients

$$\underbrace{\operatorname*{argmin}_{\theta} \sum_{i=1}^{N} \ell(y_i, h_\theta(\boldsymbol{x}_i))}_{F(\theta)}$$

change θ →
change the behavior of the classifier

How? Use Gradient Descent on $F(\theta)$!

differentiating might not always work: "… apart from the computational details"

Best Case: Optimize Empirical Risk with Gradients

$$\underset{\theta}{\operatorname{argmin}} \sum_{i=1}^{N} \ell\big(y_i, h_\theta(\boldsymbol{x}_i)\big)$$

change θ →
change the behavior of the classifier

$$\Downarrow$$

$$\nabla_\theta F = \sum_i \frac{\partial \ell\big(y_i, \hat{y} = h_\theta(\boldsymbol{x}_i)\big)}{\partial \hat{y}} \nabla_\theta h_\theta(\boldsymbol{x_i})$$

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Best Case: Optimize Empirical Risk with Gradients

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Step 1: compute the gradient of the loss wrt the predicted value

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Best Case: Optimize Empirical Risk with Gradients

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change θ →
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Step 1: compute the gradient of the loss wrt the predicted value

Step 2: compute the gradient of the predicted value wrt θ.

differentiating might not always work: "… apart from the computational details"

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Loss Functions Serve a Task

Classification Regression Clustering	Fully-supervised Semi-supervised Un-supervised	Probabilistic Generative Conditional Spectral Neural Memory-based Exemplar …
*the **task**: what* kind *of problem are you solving?*	*the **data**: amount of human input/number of labeled examples*	*the **approach**: how any data are being used*

Classification: Supervised Machine Learning

- Assigning subject categories, topics, or genres
- Spam detection
- Authorship identification
- Age/gender identification
- Language Identification
- Sentiment analysis
- …

Input:

an instance d

a fixed set of classes $C = \{c_1, c_2, \ldots, c_J\}$

A training set of m hand-labeled instances $(d_1,c_1),\ldots,(d_m,c_m)$

Output:

a learned classifier γ that maps instances to classes

γ learns to associate certain ***features*** of instances with their labels

Classification Example: Face Recognition

Class	Image
Avrim	
Avrim	
Avrim	
Avrim	

Class	Image
Tom	
Tom	
Tom	
Tom	

Courtesy from Hamed Pirsiavash

Classification Loss Function Example: 0-1 Loss

$$\ell(y, \hat{y}) = \begin{cases} 0, & \text{if } y = \hat{y} \\ 1, & \text{if } y \neq \hat{y} \end{cases}$$

Classification Loss Function Example: 0-1 Loss

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Problem 1: not differentiable wrt $\hat{y}$ (or θ)

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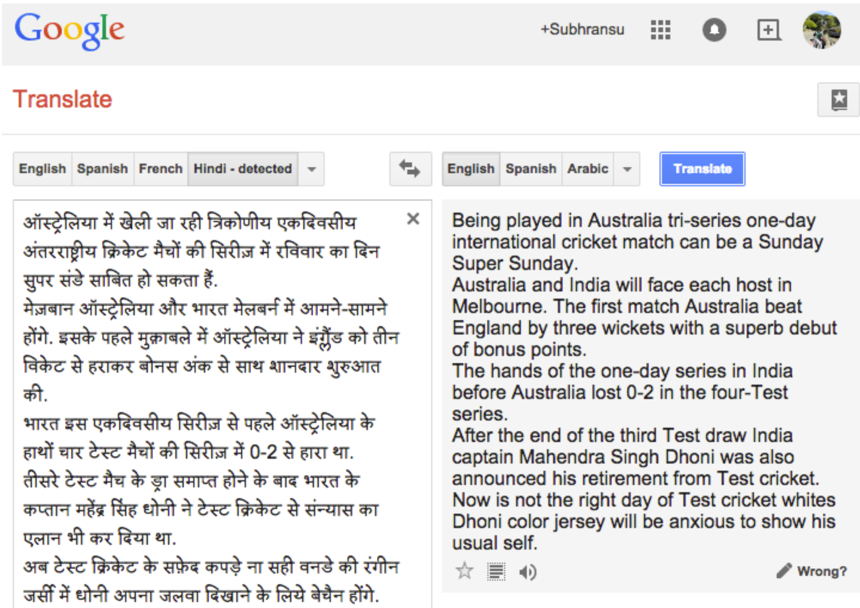
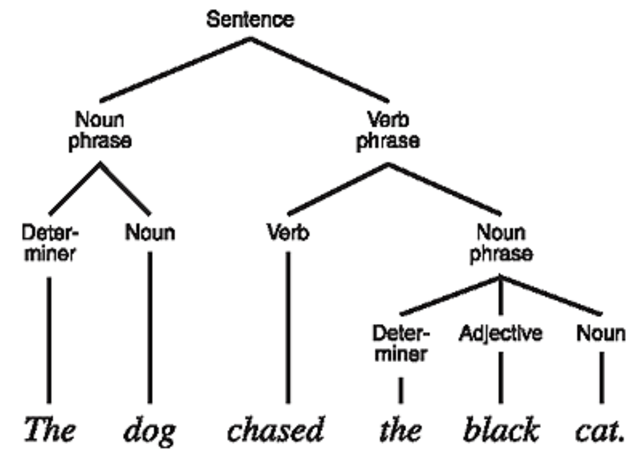
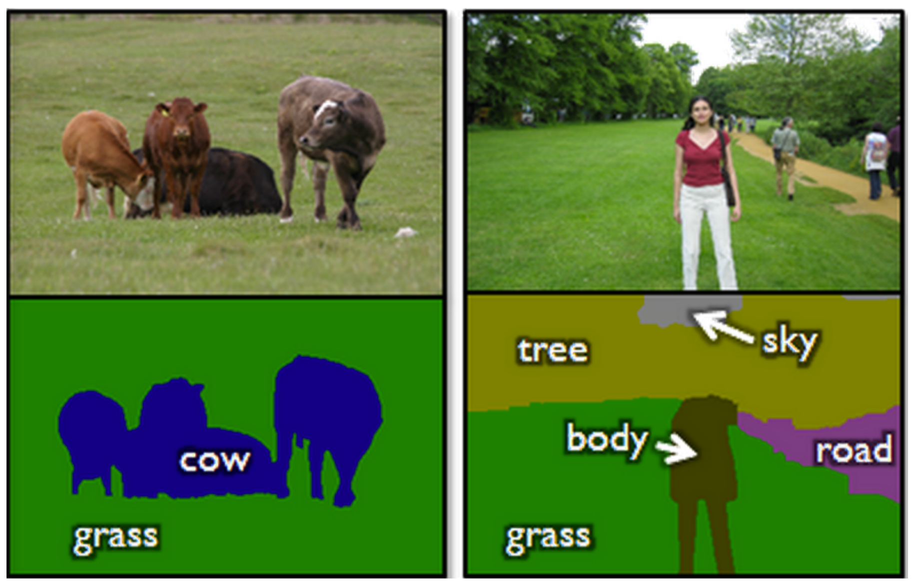
Solution 1: is the data linearly separable? Perceptron (next class) can work

Solution 2: is h(x) a conditional distribution p(y | x)? Maximize that probability (a couple classes)

Structured Classification: Sequence & Structured Prediction







Courtesy Hamed Pirsiavash

Structured Classification Loss Function Example: 0-1 Loss?

$$\ell(y, \hat{y}) = \begin{cases} 0, & \text{if } y = \hat{y} \\ 1, & \text{if } y \neq \hat{y} \end{cases}$$

Problem 1: not differentiable wrt $\hat{y}$ (or θ)

Solution 1: is the data linearly separable? Perceptron (next class) can work

Solution 2: is h(x) a conditional distribution p(y | x)? Use MAP

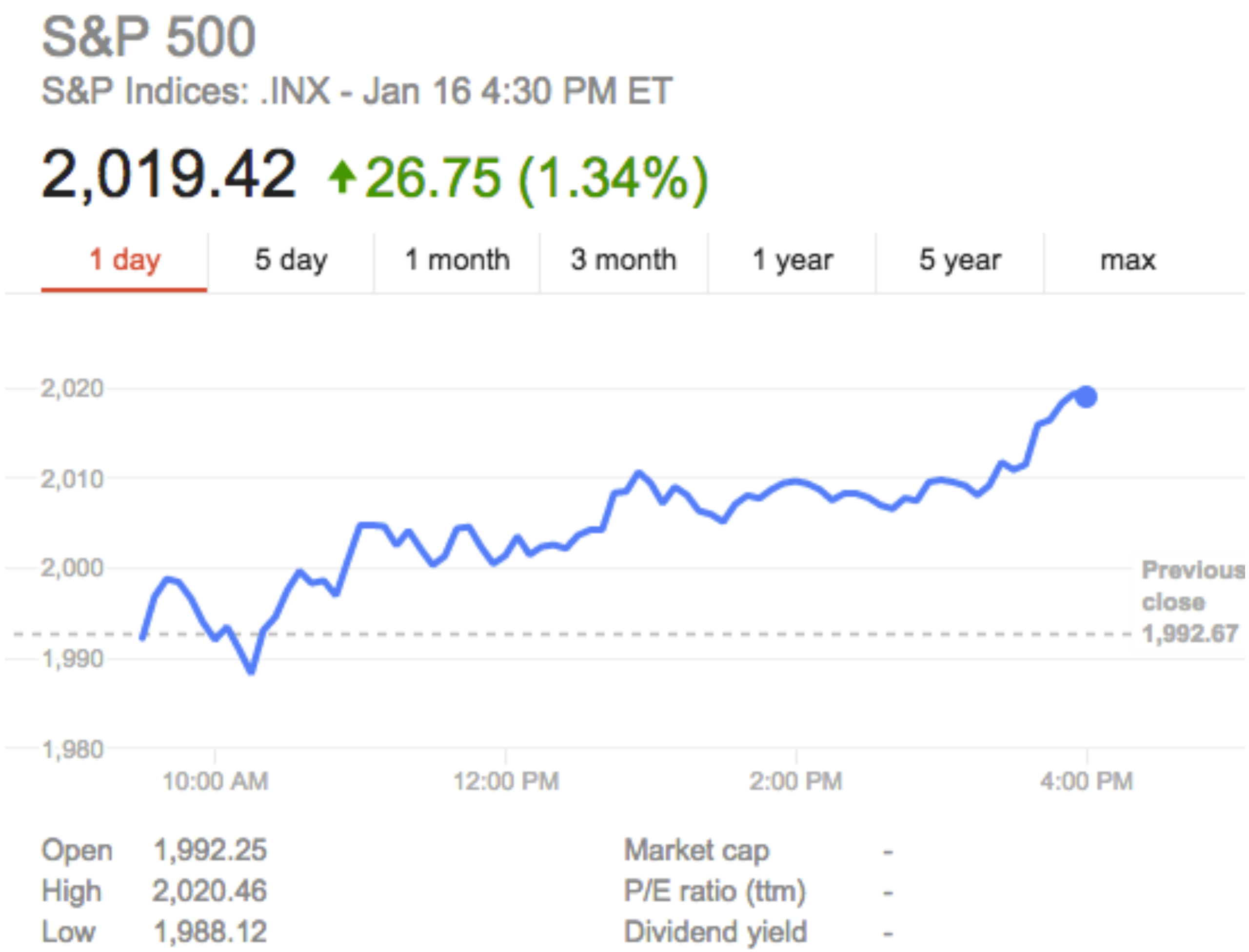
Problem 2: too strict. Structured Prediction involves many individual decisions

Solution 1: Specialize 0-1 to the structured problem at hand

Regression

Like classification, but real-valued

Regression Example: Stock Market Prediction



Courtesy Hamed Pirsiavash

Regression Loss Function Examples

squared loss/MSE
(Mean squared error)

$$\ell(y, \hat{y}) = (\mathrm{y} - \hat{y})^2$$

$\hat{y}$ is a real value → nicely differentiable (generally) ☺

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absolute loss

$$\ell(y, \hat{y}) = |y - \hat{y}|$$

Absolute value is *mostly* differentiable

Regression Loss Function Examples

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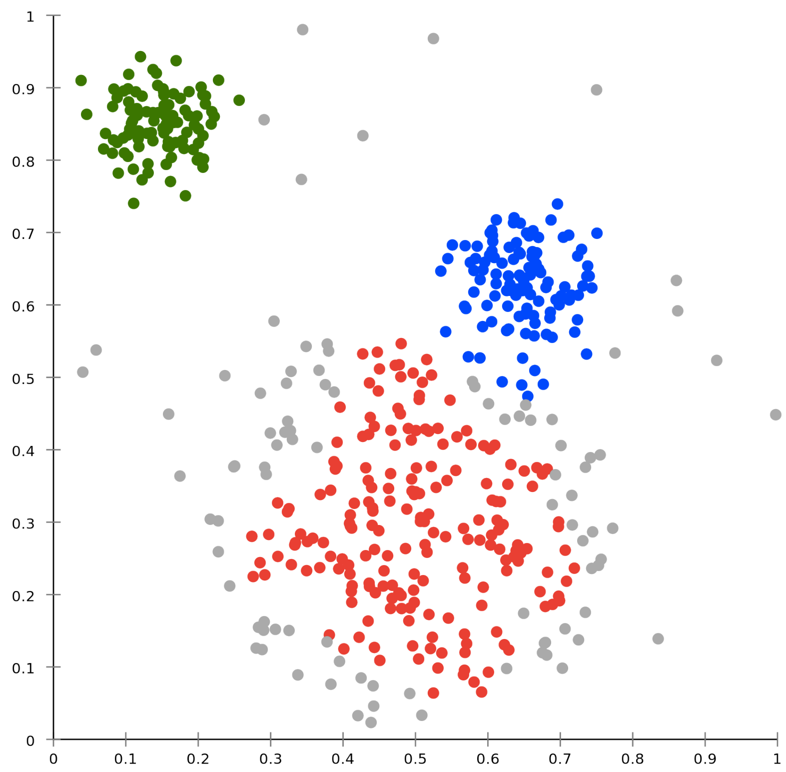
absolute loss

$$\ell(y, \hat{y}) = |y - \hat{y}|$$

Absolute value is *mostly* differentiable

These loss functions prefer different behavior in the predictions (hint: look at the gradient of each)… we'll get back to this

Unsupervised learning: Clustering



We'll return to clustering loss functions later

Courtesy Hamed Pirsiavash

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