

Midterm Exam

- Oct 20. During class (4:00 PM – 5:15 PM): Handwritten; Based on all previous lectures.
- **Closed Book Exam**
 - No notes, laptops, phones etc. – put them in your bag if you're bringing them to class.
 - Calculators are not needed (but allowed if you really need them)
 - Okay to leave answers as expressions if you don't want to calculate (example: $\frac{5}{8} * (2^6)$ is fine, final answer 40 is also fine)
- **Conceptual questions**
(we won't test you on your memorization of equations or calculation abilities)
 - multiple choice / true-false (quiz questions are a good)
 - conceptual problem solving (e.g. the Lincoln image problem in the HW)
 - design questions (e.g. if I want to find a particular pattern in an image, how would you implement it based on what we have learned in class. We might ask you for your justification, or a rough outline of your design/method.)

CMSC 472 / 672

Lecture 12

Image Transformations



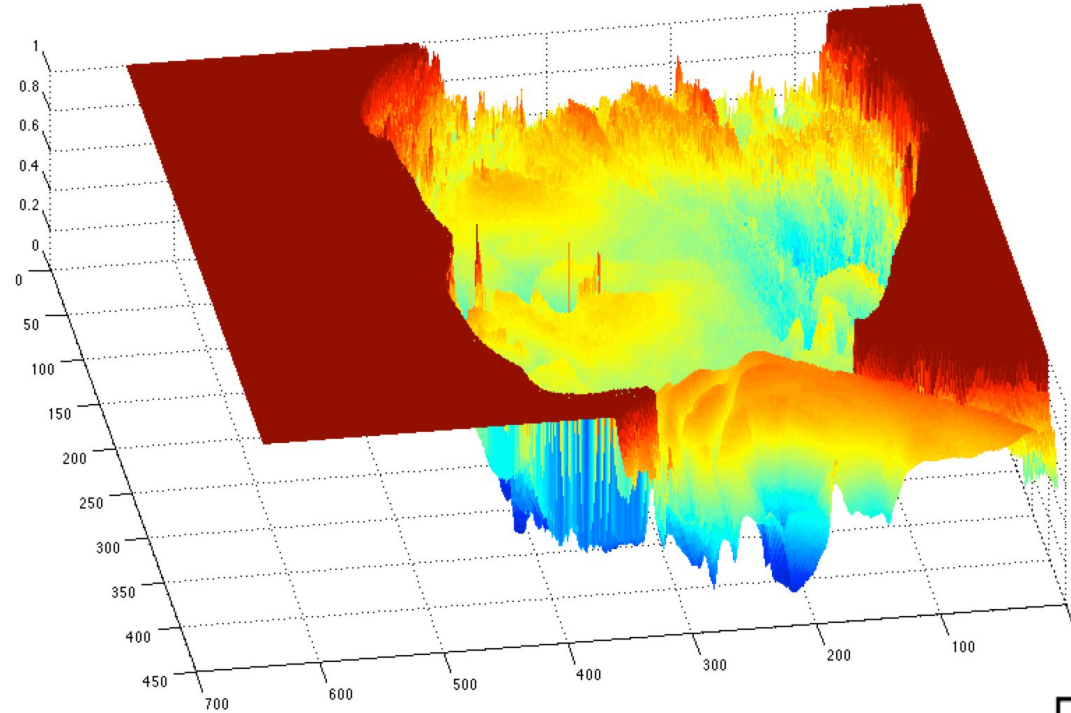
Some slides from
Yang, Jayasuriya

What is an image?



grayscale image

$$f(\mathbf{x})$$



domain $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$

A (grayscale)
image is a 2D
function.

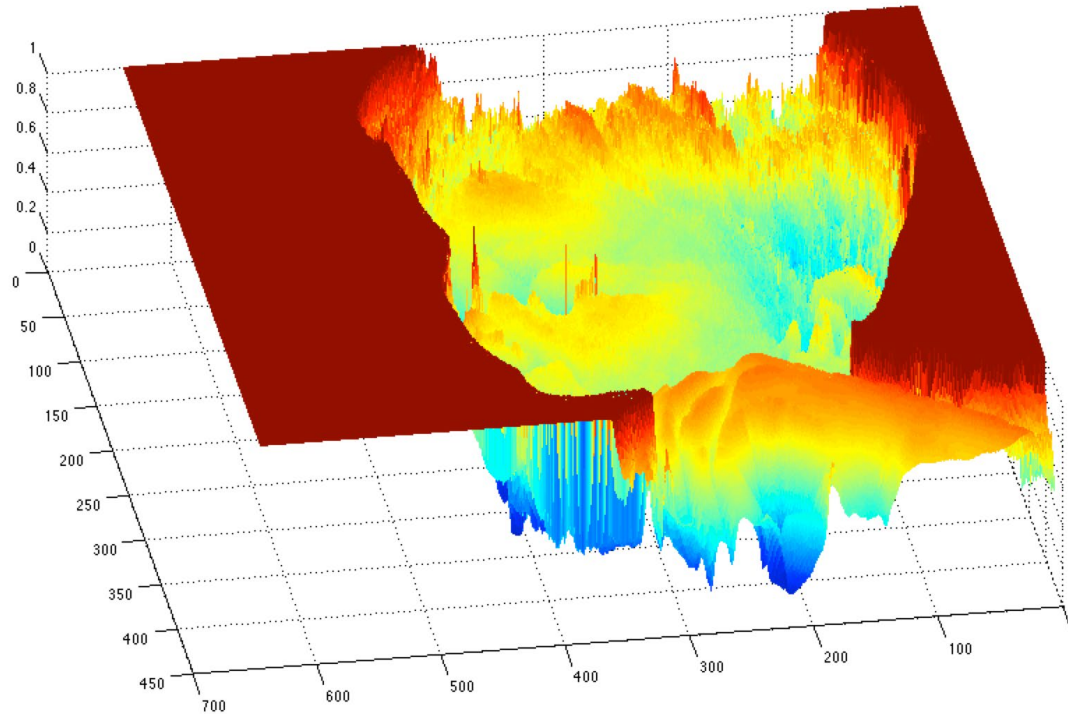
What is an image?



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$$f(\mathbf{x})$$

domain $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$



A (grayscale)
image is a 2D
function.

What is the range of
the image function f ?

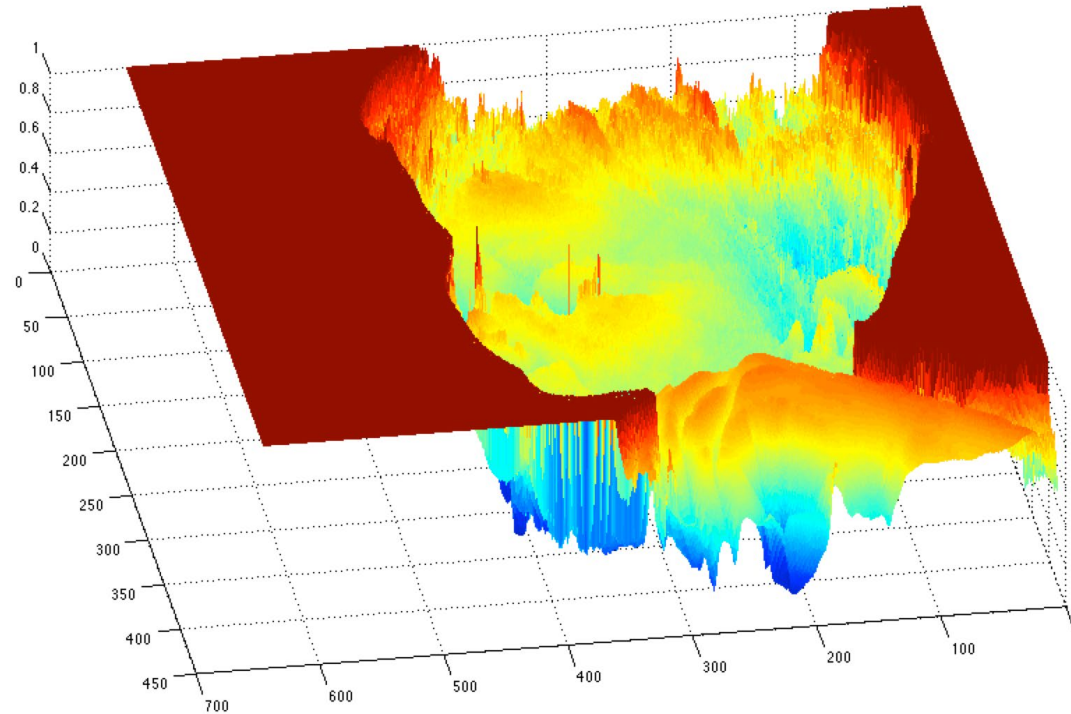
What is an image?



grayscale image

$$f(\mathbf{x})$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$



A (grayscale) image is a 2D function.

$$f(x): \text{domain} \rightarrow \text{range}$$

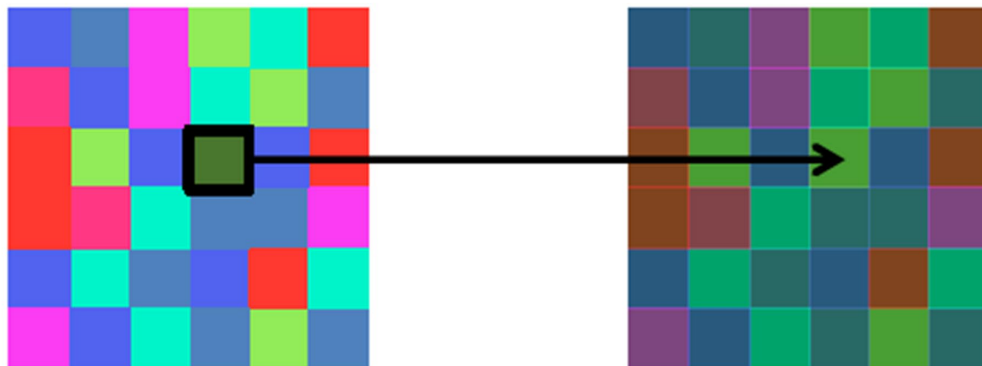
$$f(x): [0, H] \times [0, W] \rightarrow [0, 1]^3$$

What is the range of the image function f ?

RECALL

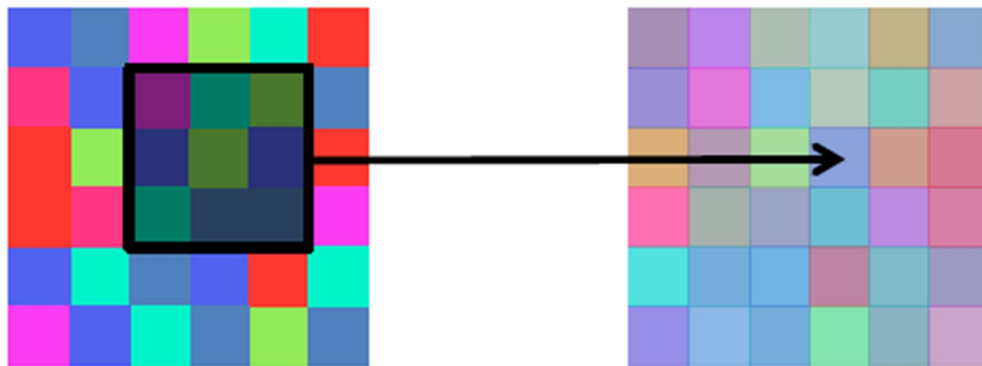
Point Processing and Image Filtering

Point Operation



point processing

Neighborhood Operation



“filtering”

What types of image transformations can we do?



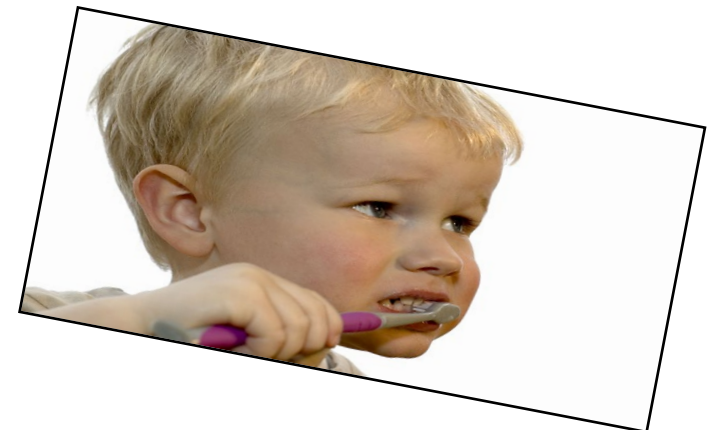
Filtering



changes pixel *values*



Warping



changes pixel **locations**

What types of image transformations can we do?

F

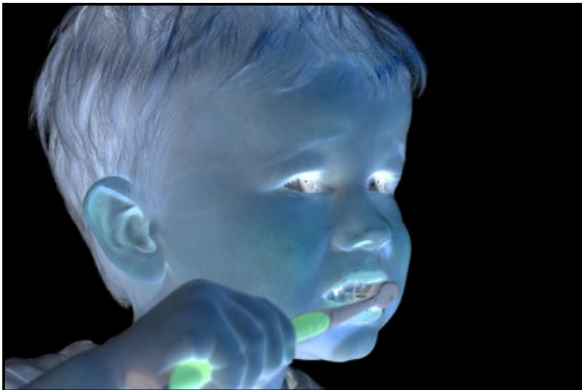


Filtering



$$G(\mathbf{x}) = h\{F(\mathbf{x})\}$$

G



changes *range* of image function

F

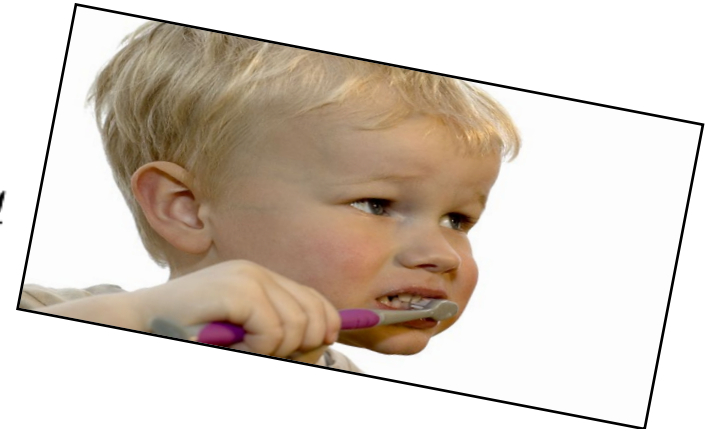


Warping



$$G(\mathbf{x}) = F(h\{\mathbf{x}\})$$

G



changes **domain** of image function

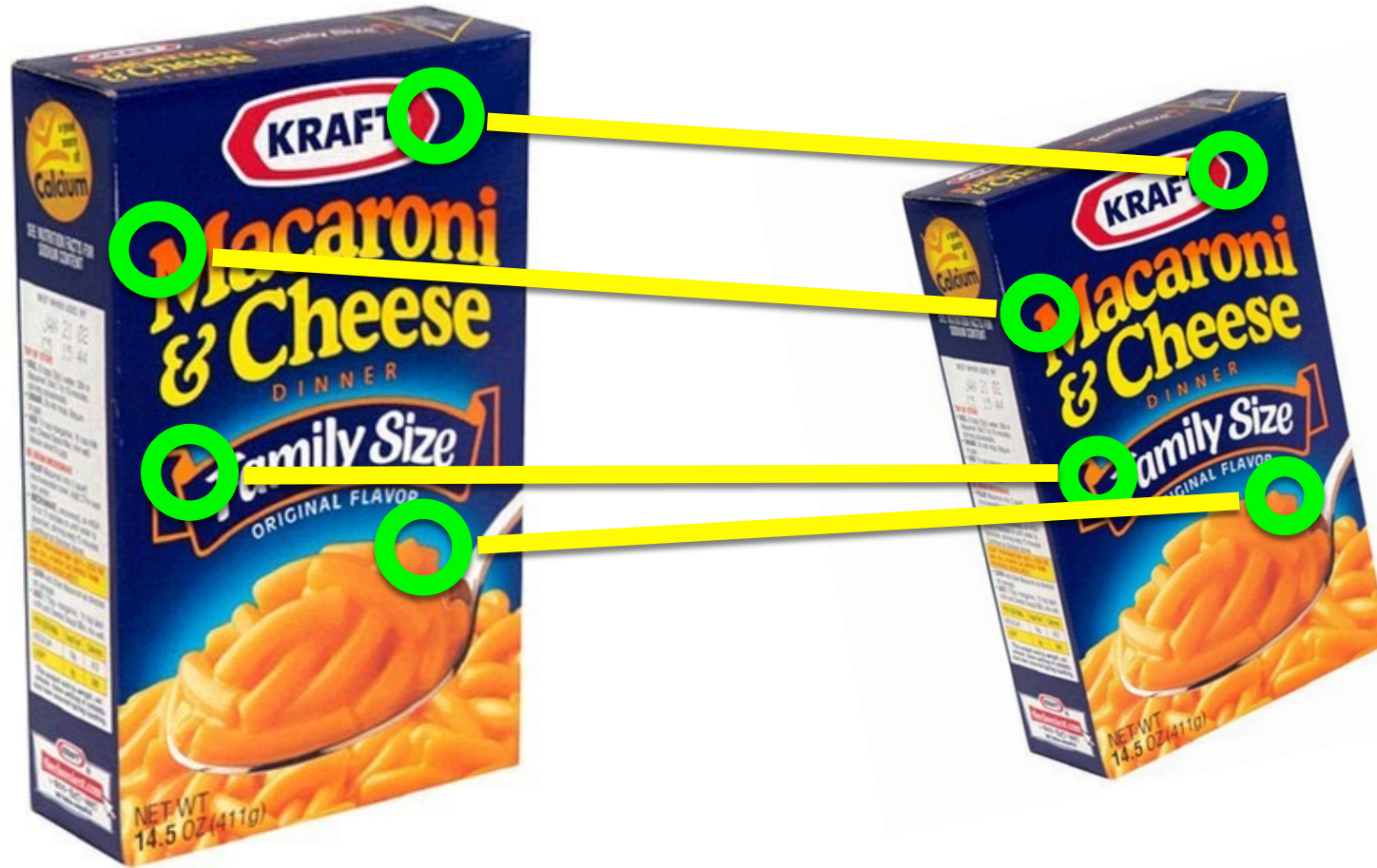
Warping example: feature matching



Warping example: feature matching



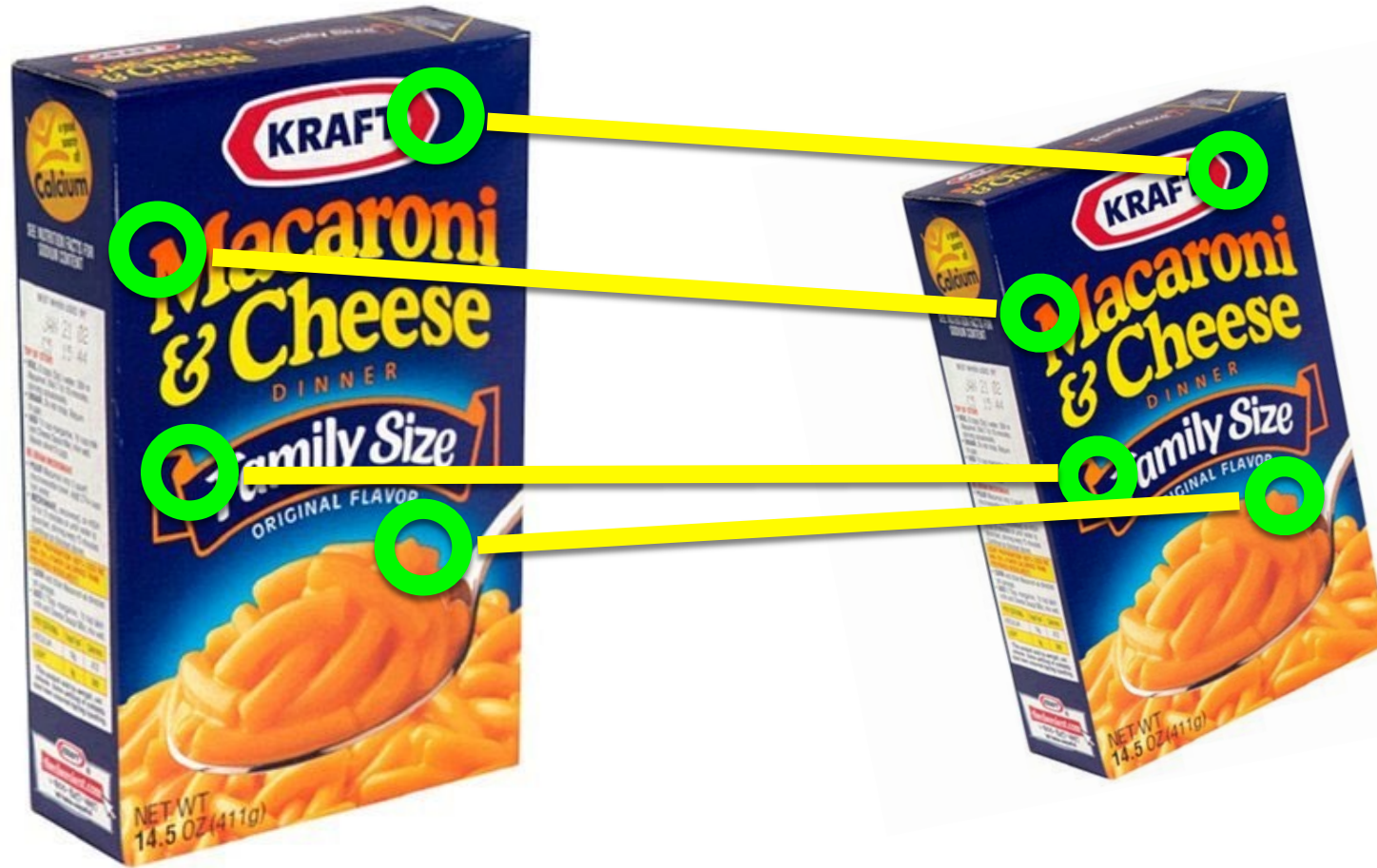
Warping example: feature matching



- object recognition
- 3D reconstruction
- augmented reality
- image stitching

How can we compute the transformation?

Warping example: feature matching



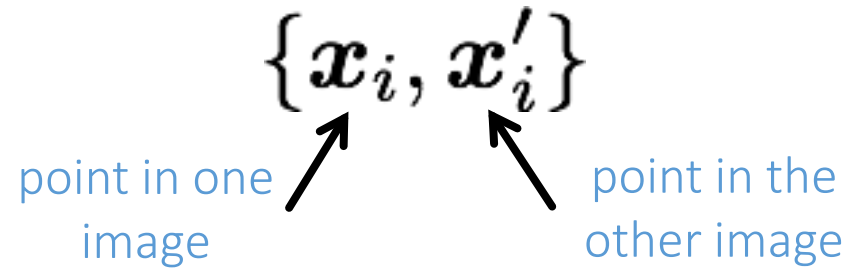
- object recognition
- 3D reconstruction
- augmented reality
- image stitching

How can we compute the transformation?

If we can, feature matching and image stitching will be easy! (HW3)

Warping example: feature matching

Given a set of matched feature points:



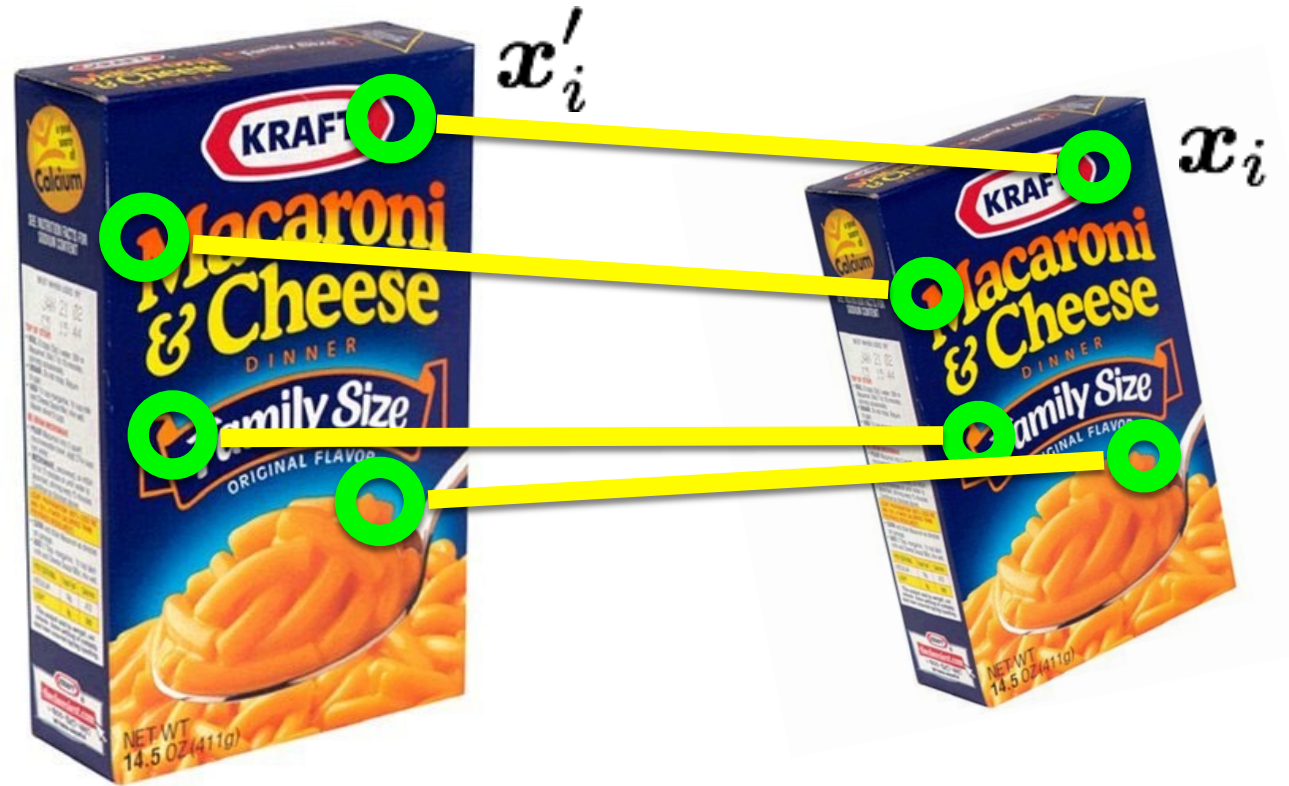
and a transformation:

$$x' = f(x; p)$$

transformation function parameters

find the best estimate of the parameters

p



What kind of transformation functions f are there?

2D transformations

What kind of transformation functions f are there?

2D transformations



translation



rotation



aspect



affine



perspective

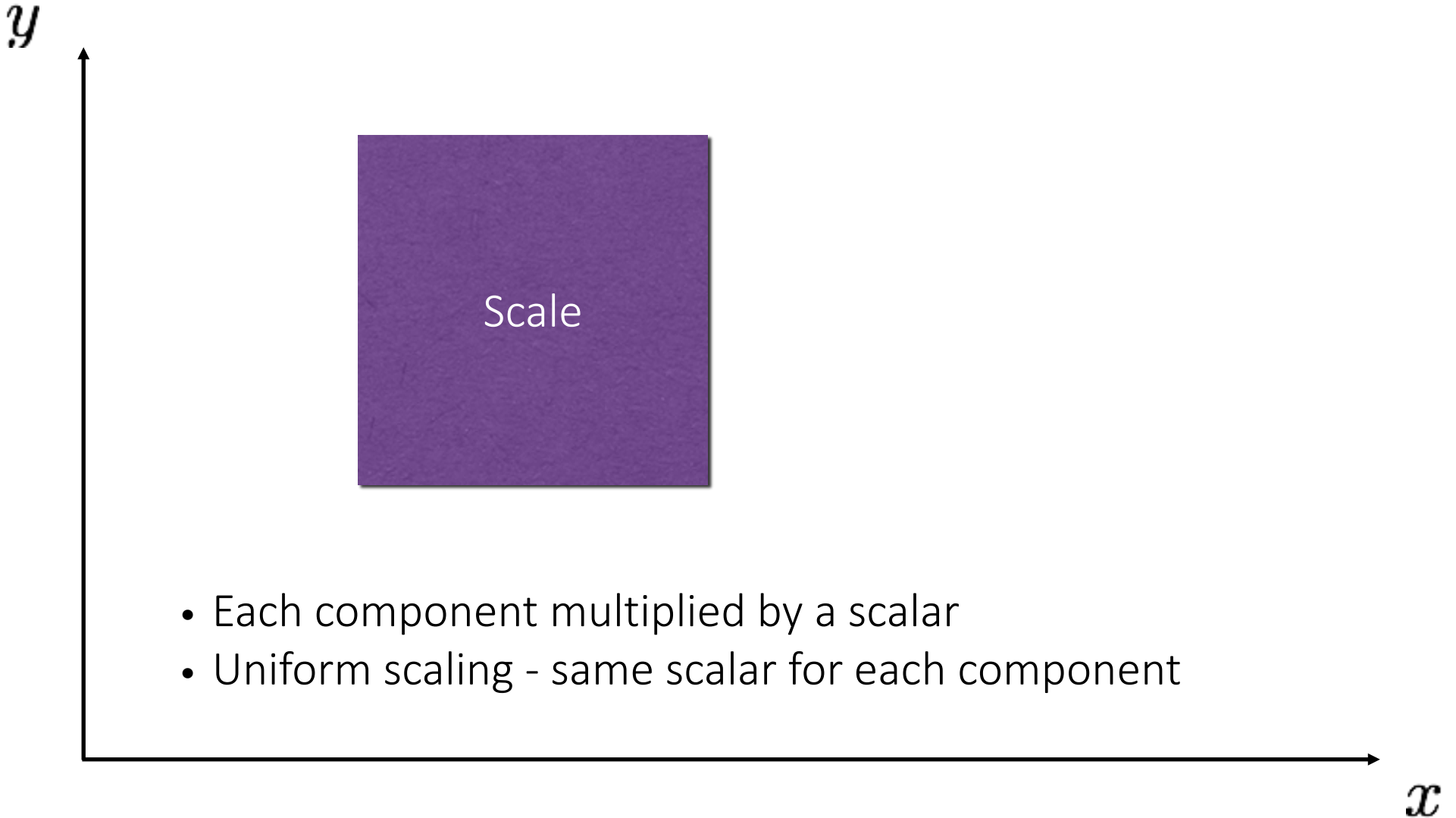


cylindrical

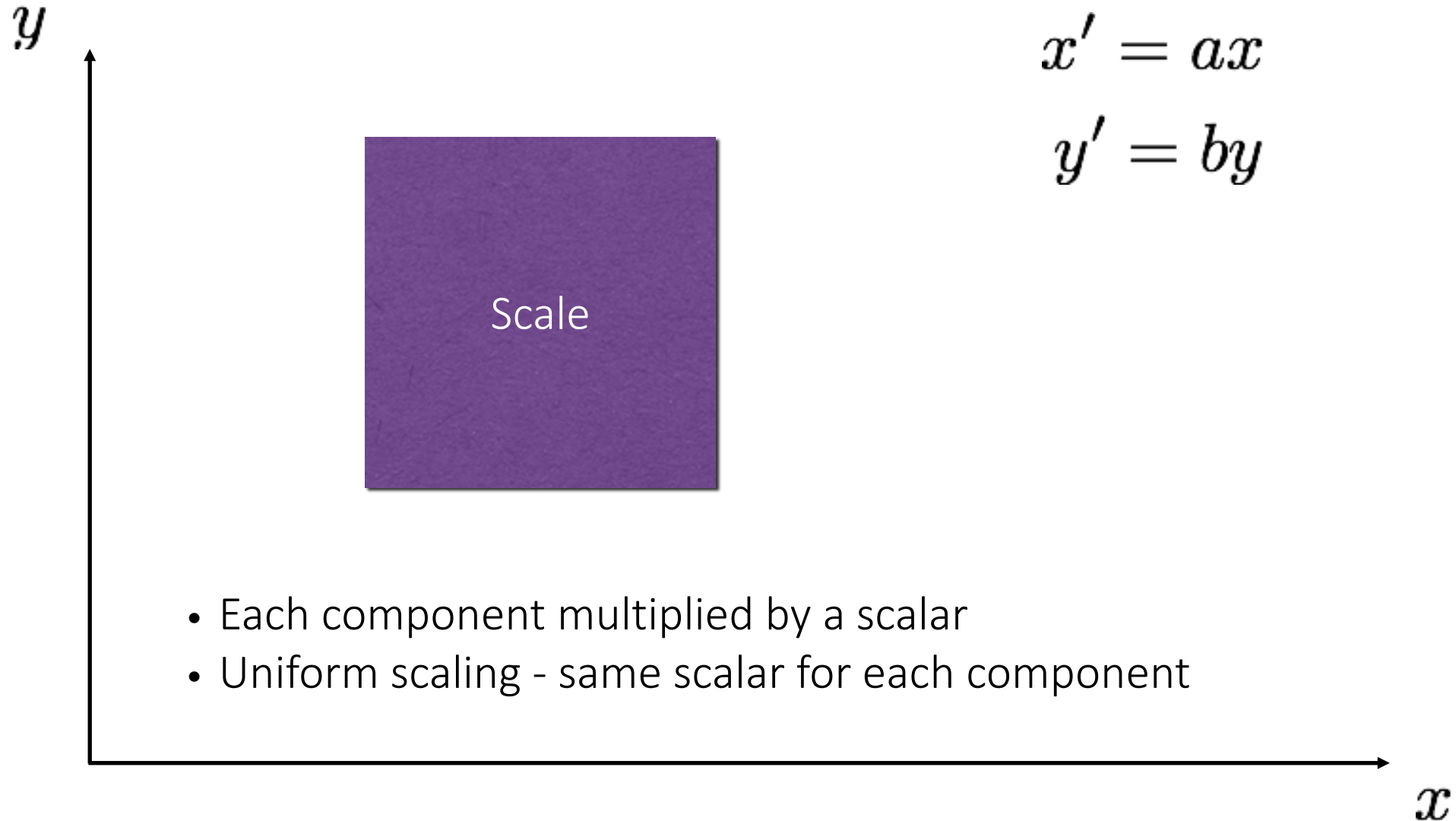
2D planar transformations



2D planar transformations



2D planar transformations



2D planar transformations

y



$$x' = ax$$

$$y' = by$$

matrix representation of scaling:

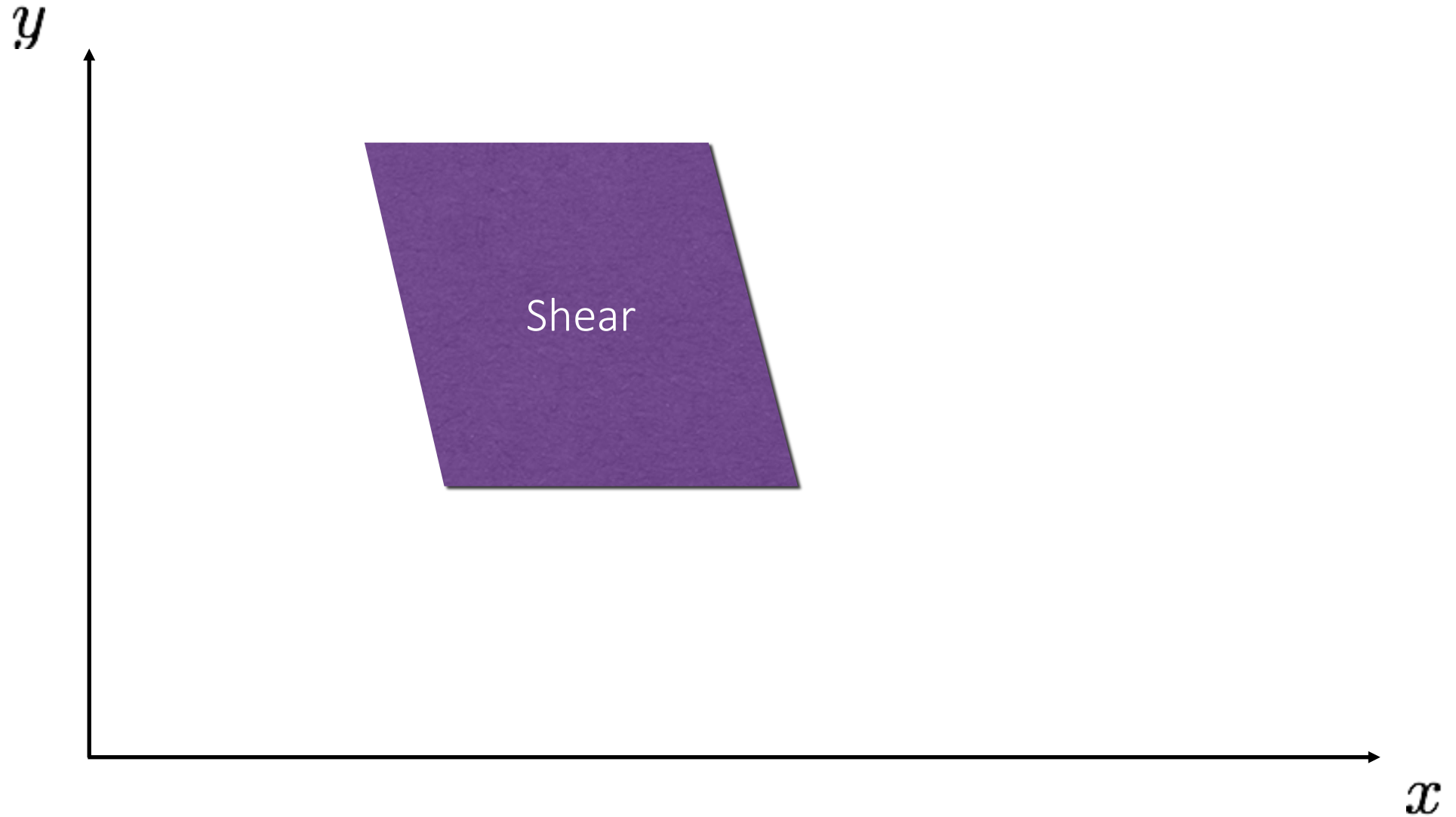
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}}_{\text{scaling matrix } S} \begin{bmatrix} x \\ y \end{bmatrix}$$

scaling matrix S

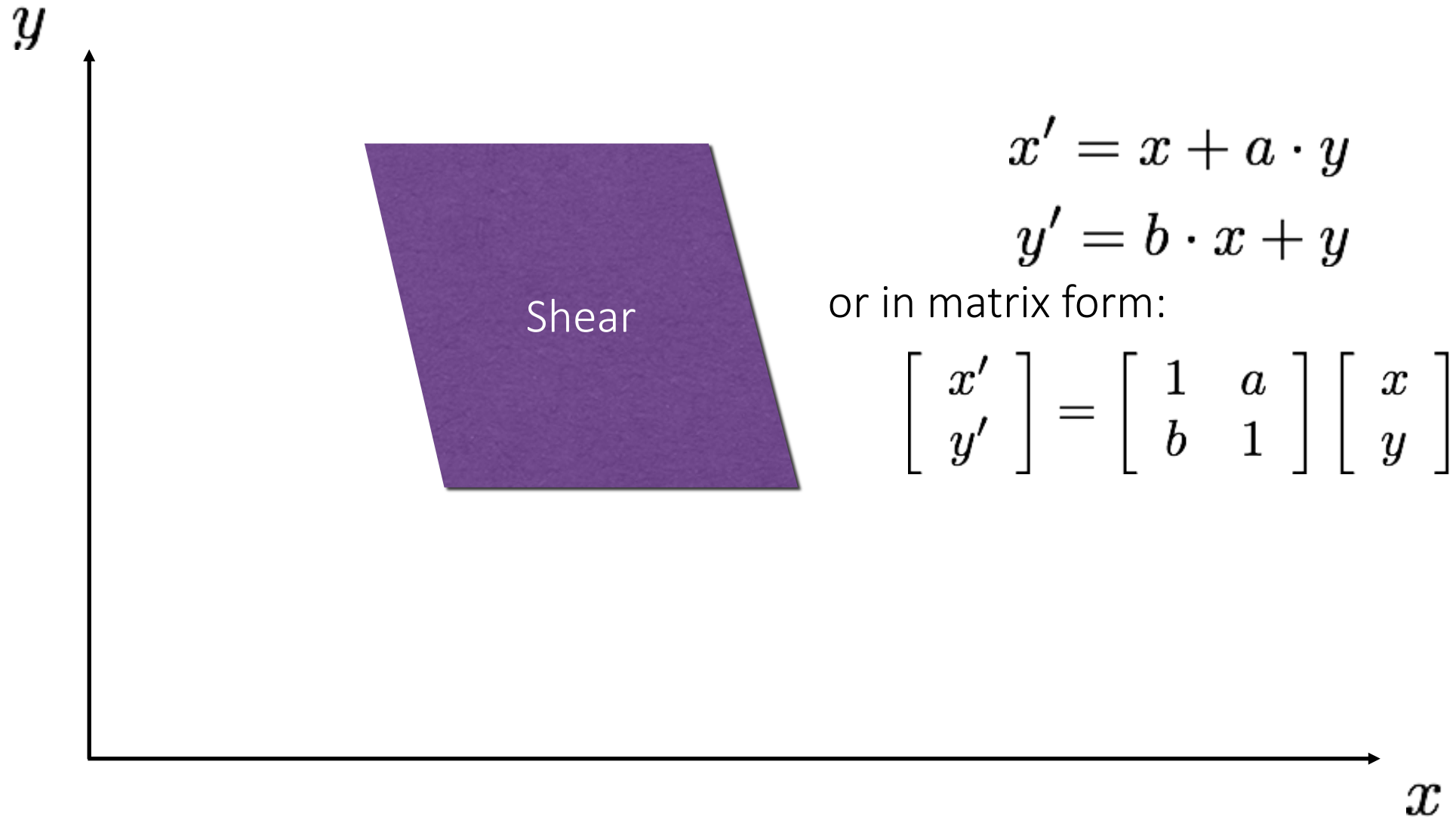
- Each component multiplied by a scalar
- Uniform scaling - same scalar for each component

x

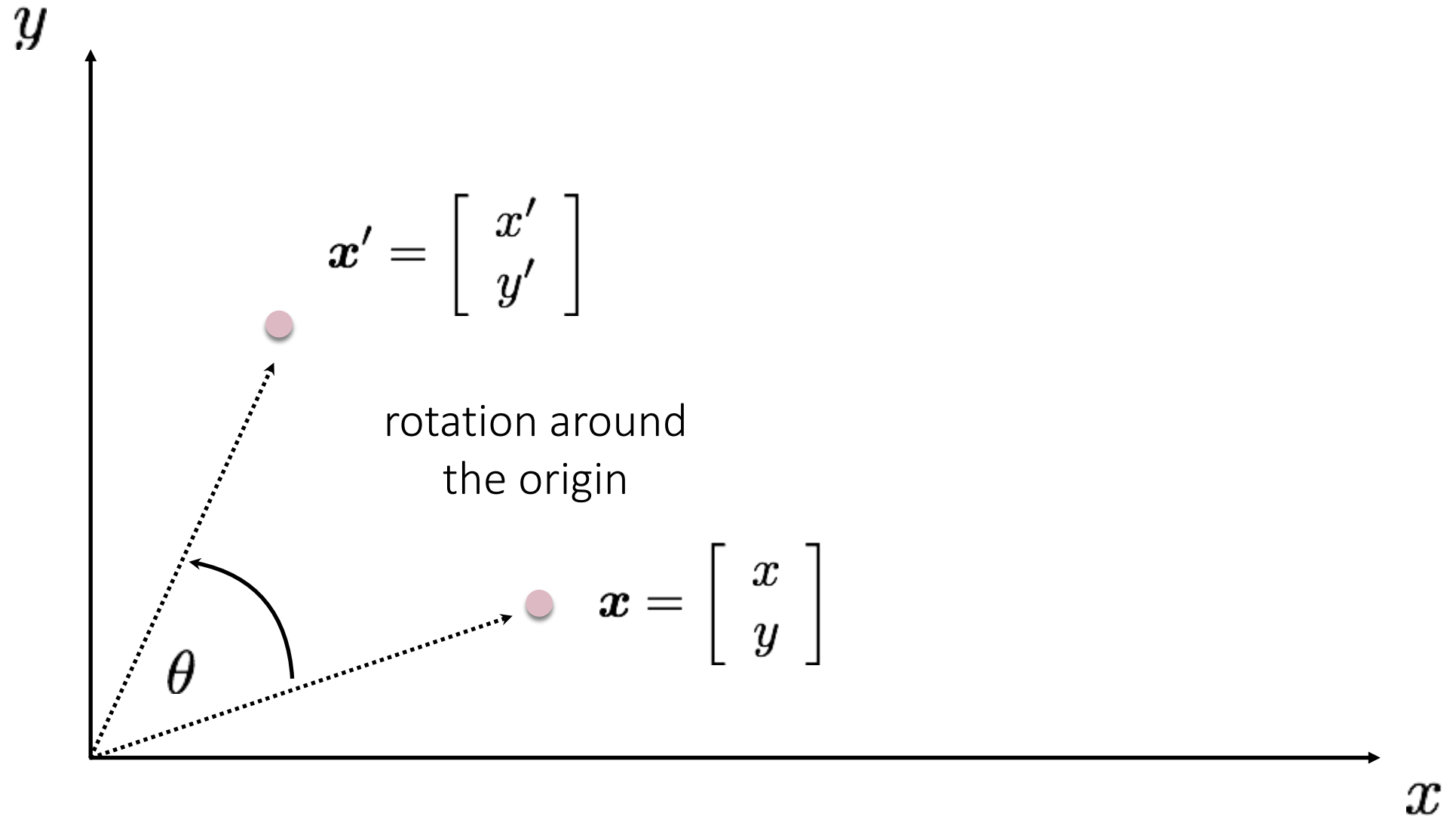
2D planar transformations



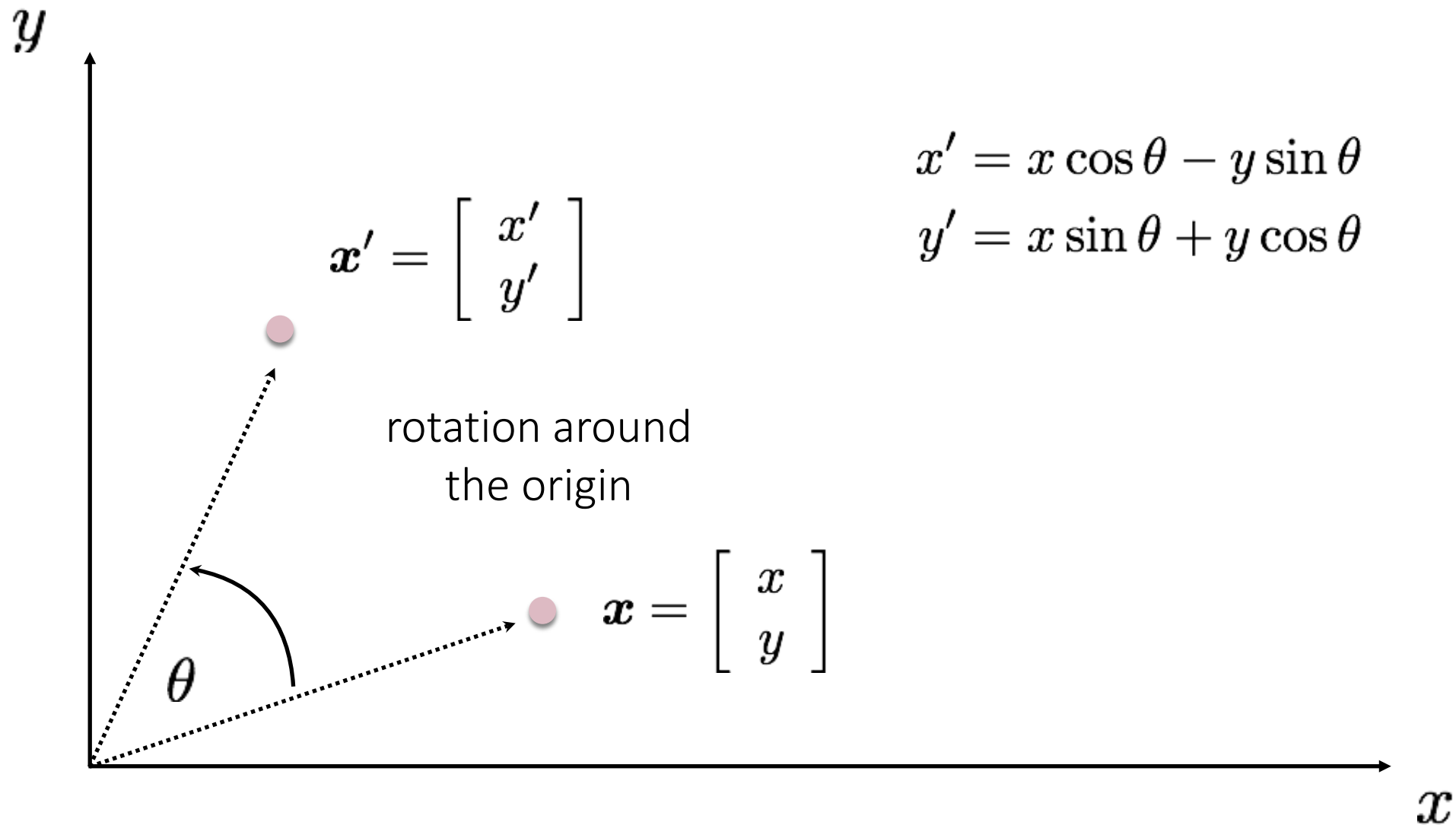
2D planar transformations



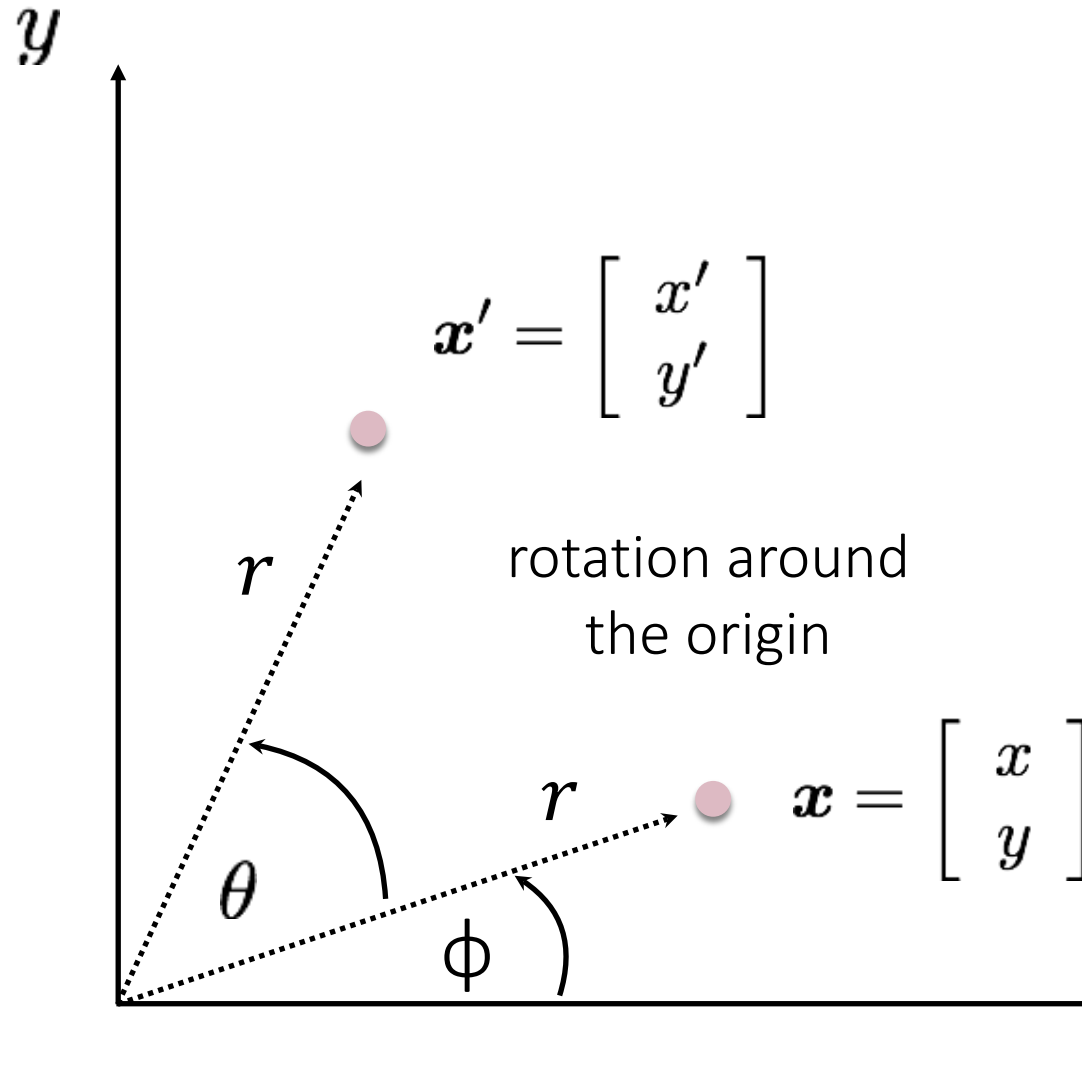
2D planar transformations



2D planar transformations



2D planar transformations



Polar coordinates...

$$x = r \cos(\phi)$$

$$y = r \sin(\phi)$$

$$x' = r \cos(\phi + \theta)$$

$$y' = r \sin(\phi + \theta)$$

Trigonometric Identity...

$$x' = r \cos(\phi) \cos(\theta) - r \sin(\phi) \sin(\theta)$$

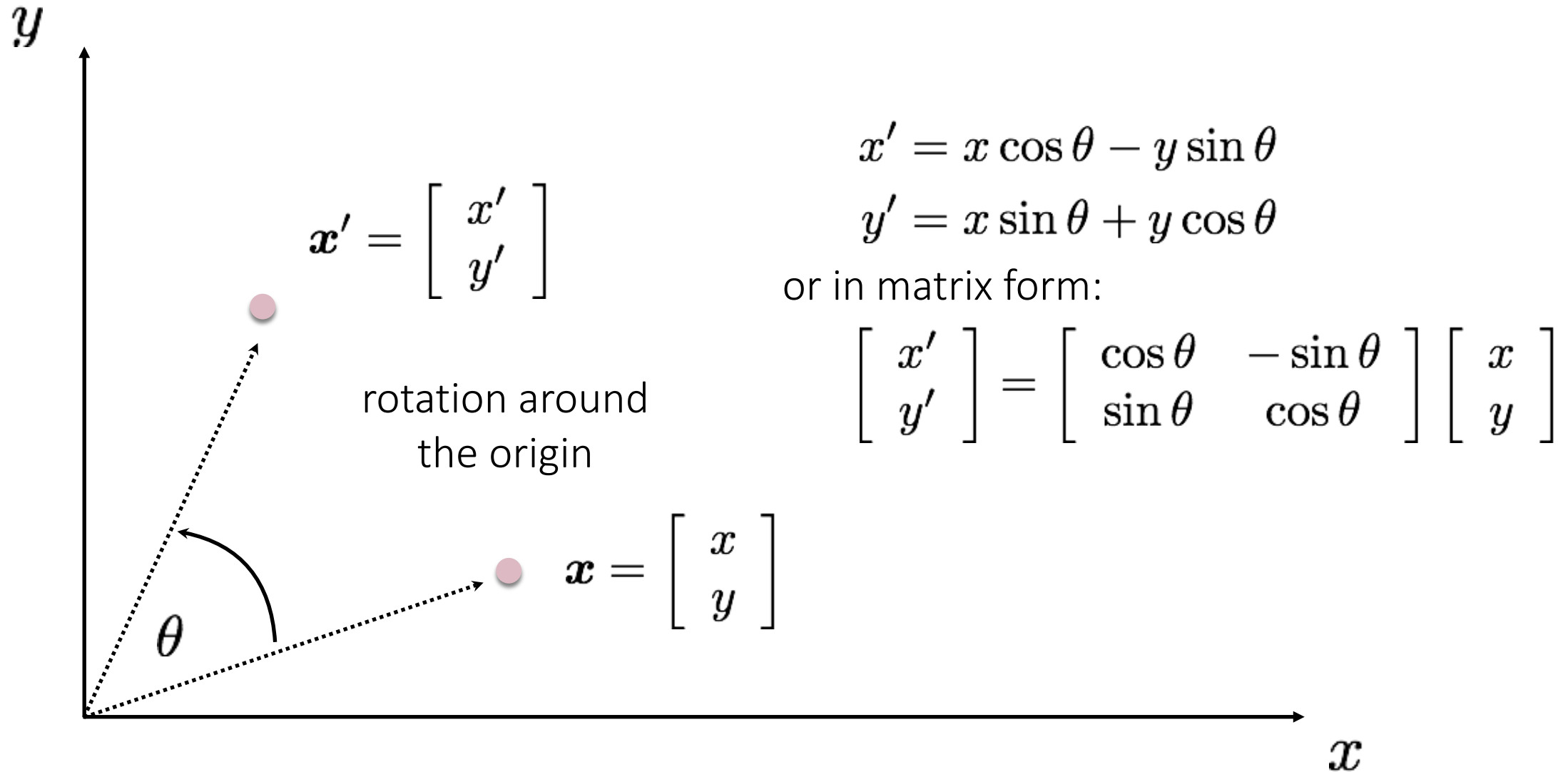
$$y' = r \sin(\phi) \cos(\theta) + r \cos(\phi) \sin(\theta)$$

Substitute...

$$x' = x \cos(\theta) - y \sin(\theta)$$

$$y' = x \sin(\theta) + y \cos(\theta)$$

2D planar transformations



2D planar and linear transformations

$$\boldsymbol{x}' = f(\boldsymbol{x}; p)$$



$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \boldsymbol{M} \begin{bmatrix} x \\ y \end{bmatrix}$$

parameters p

point $\boldsymbol{x}$

2D planar and linear transformations

Scale

$$\mathbf{M} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$$

Flip across y

$$\mathbf{M} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Rotate

$$\mathbf{M} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

Flip across origin

$$\mathbf{M} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

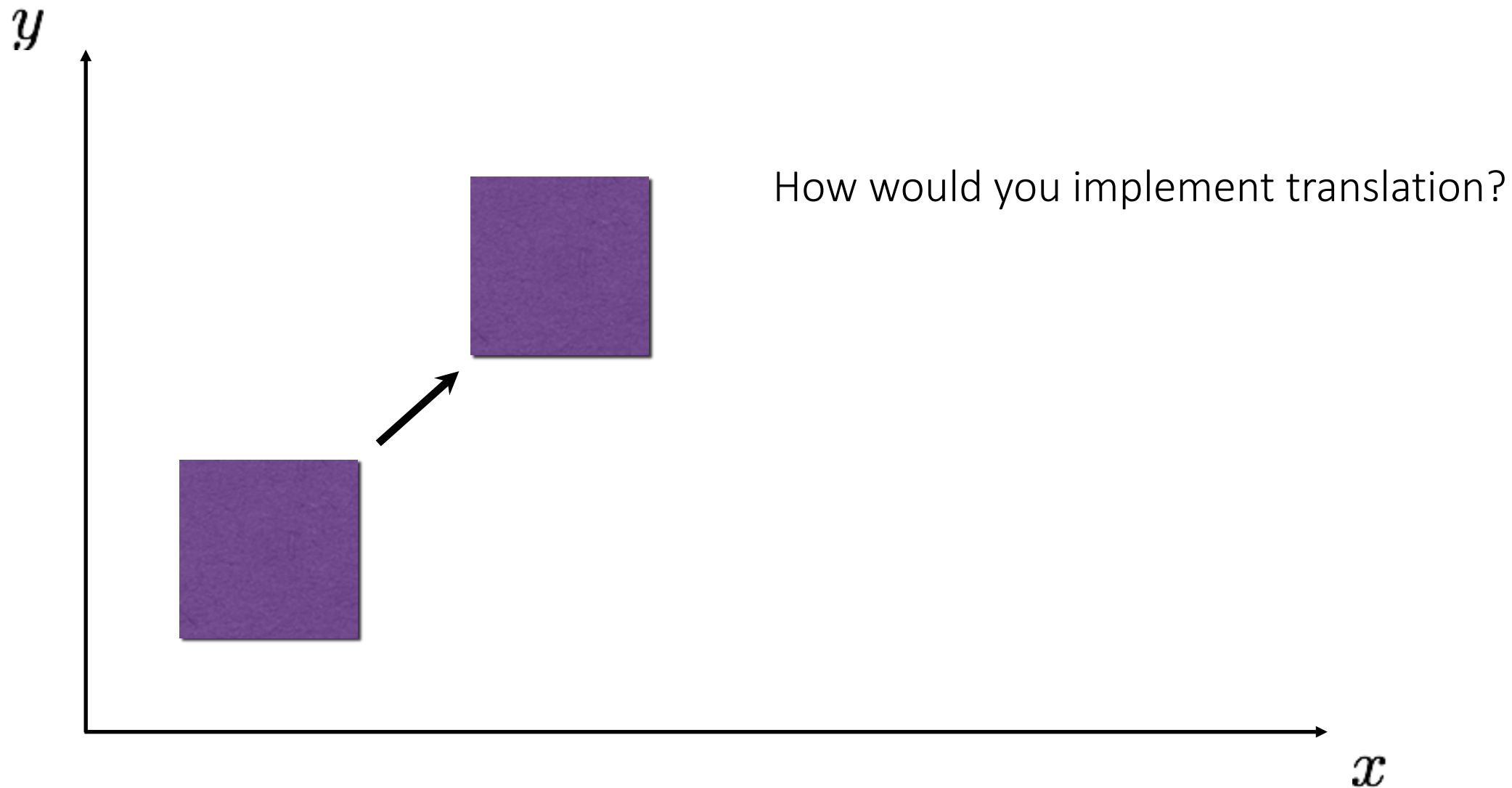
Shear

$$\mathbf{M} = \begin{bmatrix} 1 & s_x \\ s_y & 1 \end{bmatrix}$$

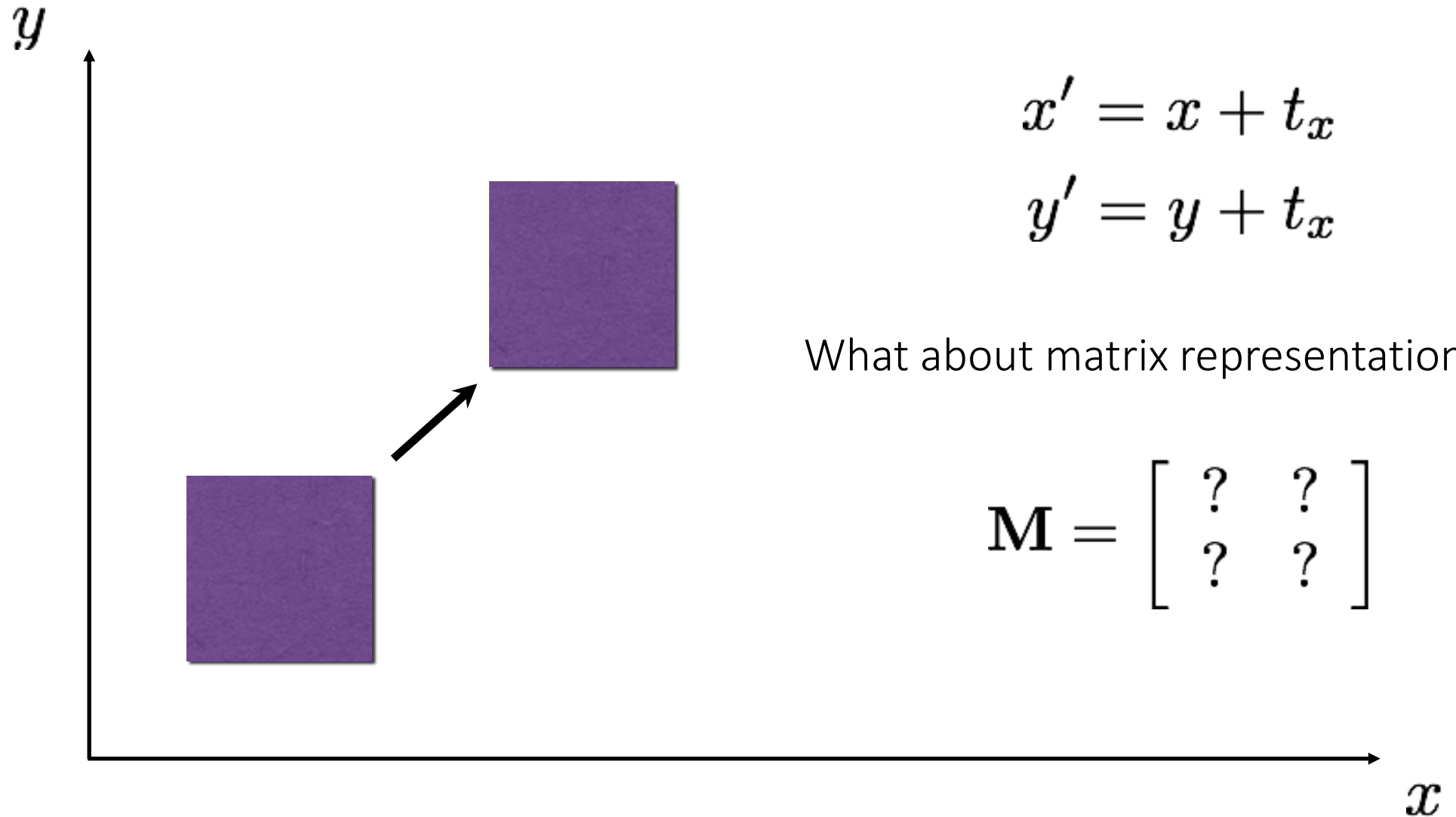
Identity

$$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

2D translation

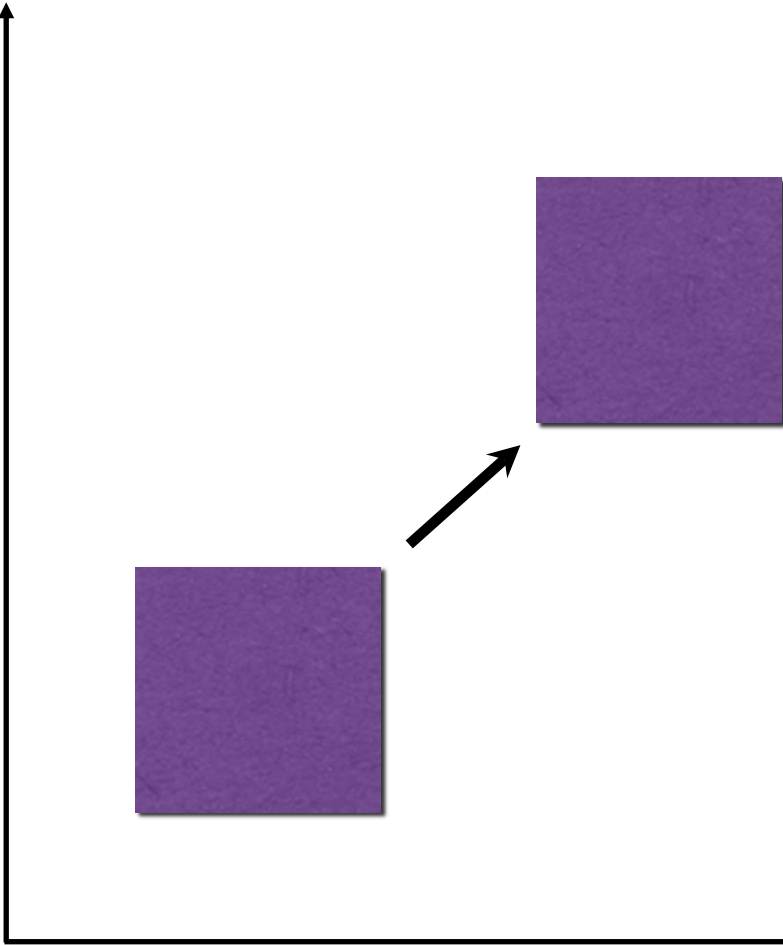


2D translation



2D translation

y



$$x' = x + t_x$$

$$y' = y + t_y$$

What about matrix representation?

Not possible.

x

Projective geometry 101

Homogeneous coordinates

heterogeneous
coordinates

homogeneous
coordinates

$$\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

add a 1 here



- Represent 2D point with a 3D vector

Homogeneous coordinates

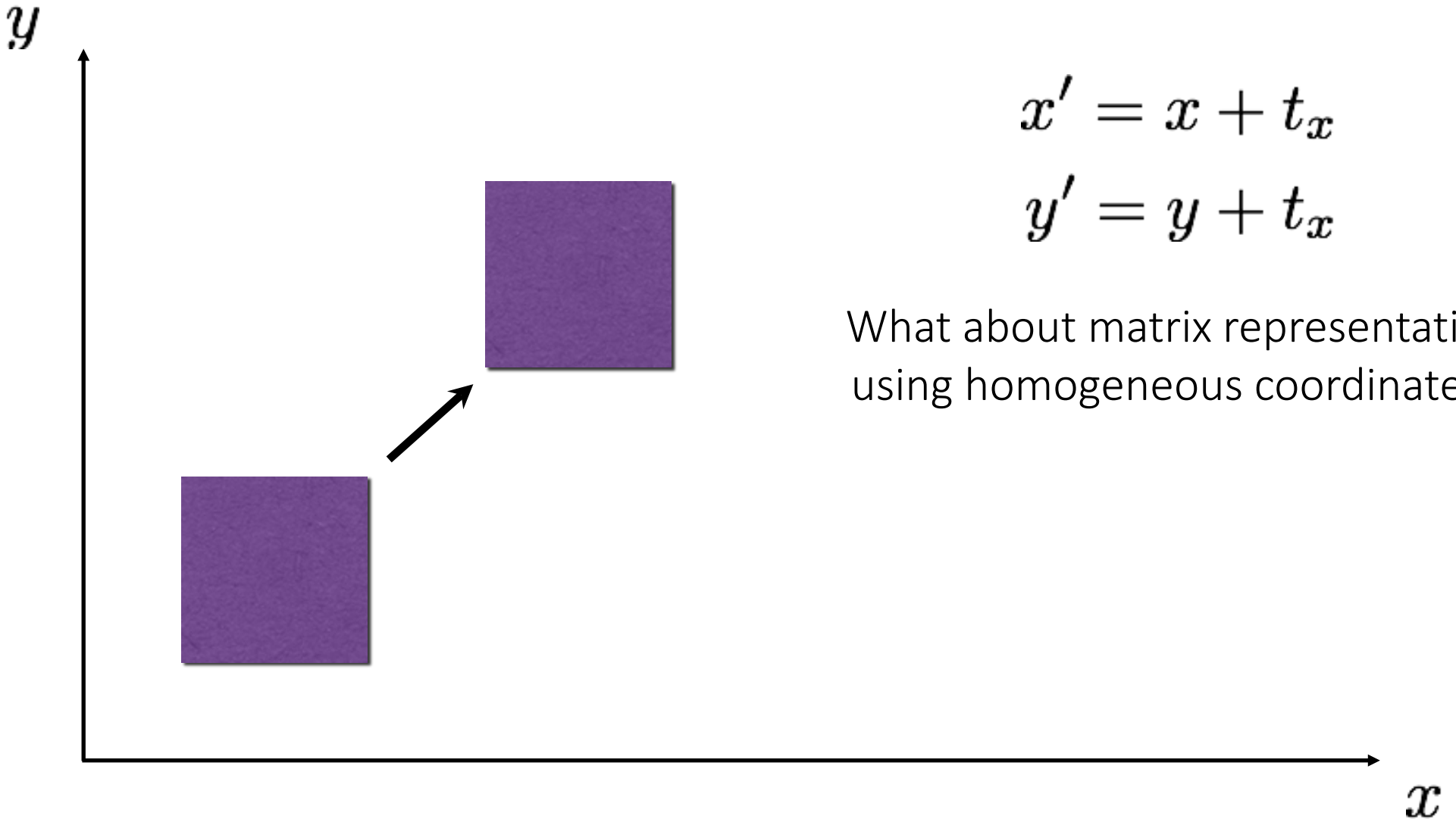
heterogeneous
coordinates

homogeneous
coordinates

$$\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} ax \\ ay \\ a \end{bmatrix}$$

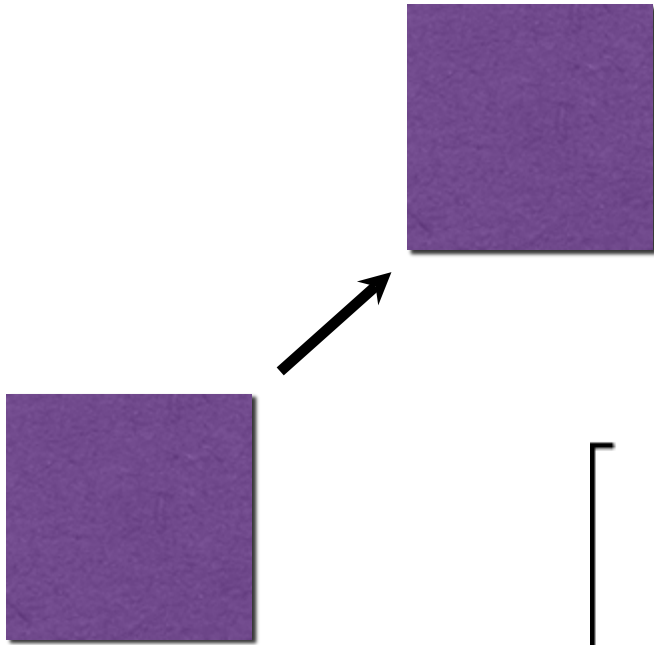
- Represent 2D point with a 3D vector
- 3D vectors are only defined up to scale

2D translation



2D translation

y



$$x' = x + t_x$$

$$y' = y + t_y$$

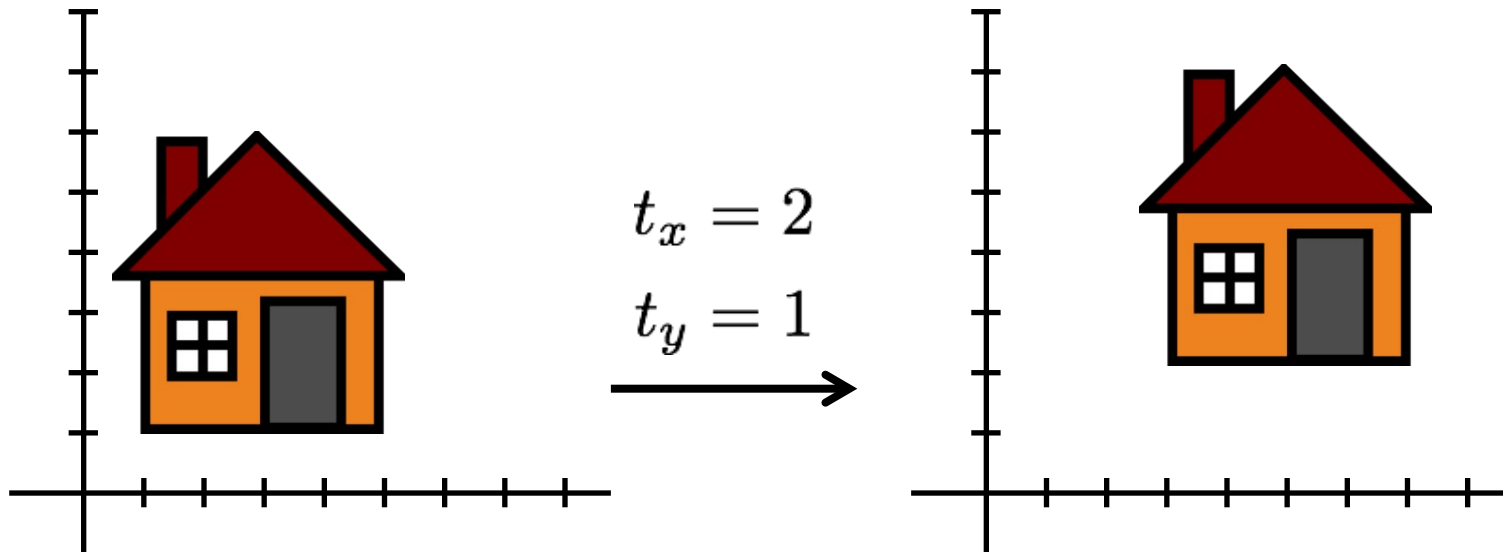
What about matrix representation
using homogenous coordinates?

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \mathbf{M} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

x

2D translation using homogeneous coordinates

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix}$$



Homogeneous coordinates

Conversion:

- heterogeneous $\rightarrow$ homogeneous

$$\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- homogeneous $\rightarrow$ heterogeneous

$$\begin{bmatrix} x \\ y \\ w \end{bmatrix} \Rightarrow \begin{bmatrix} x/w \\ y/w \end{bmatrix}$$

- scale invariance

$$\begin{bmatrix} x & y & w \end{bmatrix}^{\top} = \lambda \begin{bmatrix} x & y & w \end{bmatrix}^{\top}$$

Special points:

- point at infinity

$$\begin{bmatrix} x & y & 0 \end{bmatrix}$$

- undefined

$$\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

Projective geometry

image point in
pixel coordinates

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

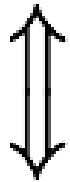
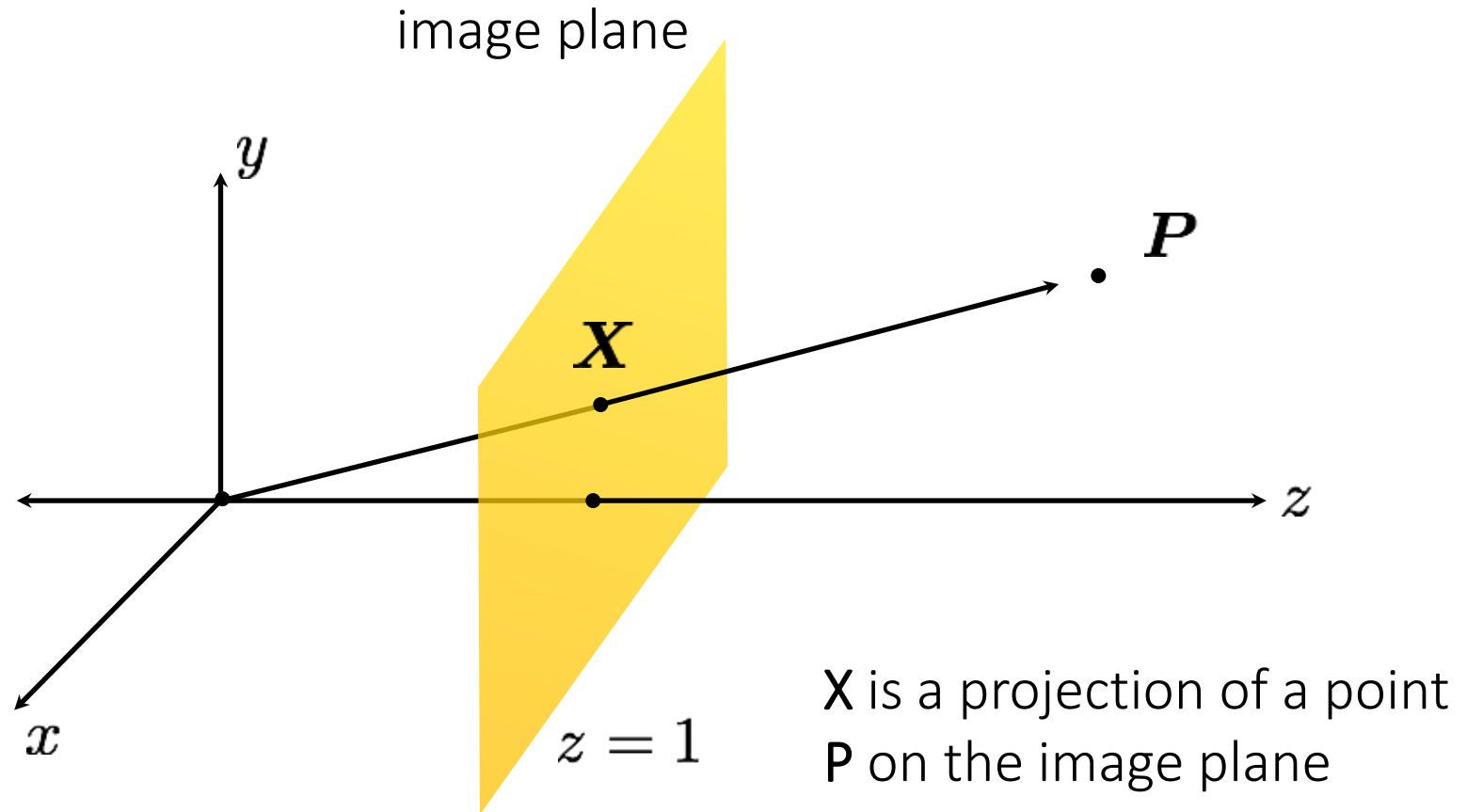


image point in
homogeneous
coordinates

$$\mathbf{X} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$



What does scaling $\mathbf{X}$ correspond to?

Transformations in projective geometry

2D transformations in heterogeneous coordinates

Re-write these transformations as 3x3 matrices:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

translation

$$\begin{bmatrix} \mathbf{x}' \\ \mathbf{y}' \\ 1 \end{bmatrix} = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \\ 1 \end{bmatrix}$$

scaling

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

rotation

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

shearing

2D transformations in heterogeneous coordinates

Re-write these transformations as 3x3 matrices:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

translation

$$\begin{bmatrix} \mathbf{x}' \\ \mathbf{y}' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \\ 1 \end{bmatrix}$$

scaling

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} & & \\ & ? & \\ & & \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

rotation

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} & & \\ & ? & \\ & & \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

shearing

2D transformations in heterogeneous coordinates

Re-write these transformations as 3x3 matrices:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

translation

$$\begin{bmatrix} \mathbf{x}' \\ \mathbf{y}' \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{s}_x & 0 & 0 \\ 0 & \mathbf{s}_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \\ 1 \end{bmatrix}$$

scaling

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} & & \\ & ? & \\ & & \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

rotation

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & \beta_x & 0 \\ \beta_y & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

shearing

2D transformations in heterogeneous coordinates

Re-write these transformations as 3x3 matrices:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

translation

$$\begin{bmatrix} \mathbf{x}' \\ \mathbf{y}' \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{s}_x & 0 & 0 \\ 0 & \mathbf{s}_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \\ 1 \end{bmatrix}$$

scaling

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos \Theta & -\sin \Theta & 0 \\ \sin \Theta & \cos \Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

rotation

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & \beta_x & 0 \\ \beta_y & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

shearing

Matrix composition

Transformations can be combined by matrix multiplication:

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \left(\begin{bmatrix} 1 & 0 & tx \\ 0 & 1 & ty \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \Theta & -\sin \Theta & 0 \\ \sin \Theta & \cos \Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} sx & 0 & 0 \\ 0 & sy & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

$\mathbf{p}' = \quad ? \quad ? \quad ? \quad \mathbf{p}$

Matrix composition

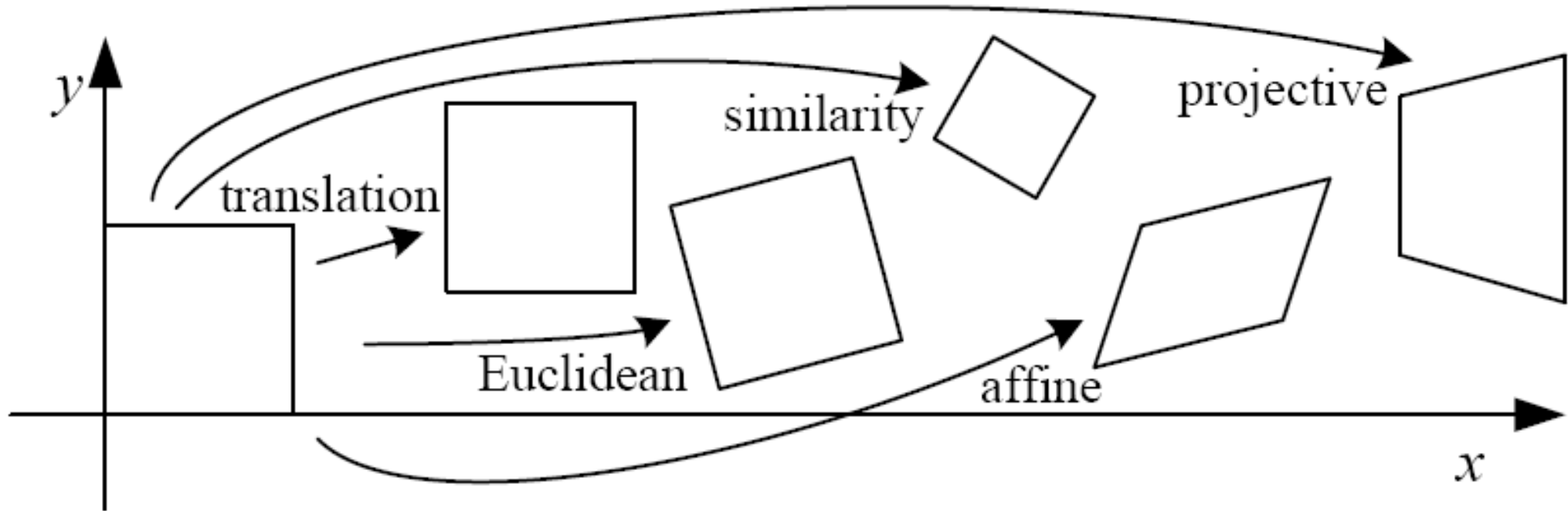
Transformations can be combined by matrix multiplication:

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \left(\begin{bmatrix} 1 & 0 & tx \\ 0 & 1 & ty \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \Theta & -\sin \Theta & 0 \\ \sin \Theta & \cos \Theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} sx & 0 & 0 \\ 0 & sy & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

$$\mathbf{p}' = \text{translation}(t_x, t_y) \quad \text{rotation}(\theta) \quad \text{scale}(s, s) \quad \mathbf{p}$$

Classification of 2D transformations

Classification of 2D transformations



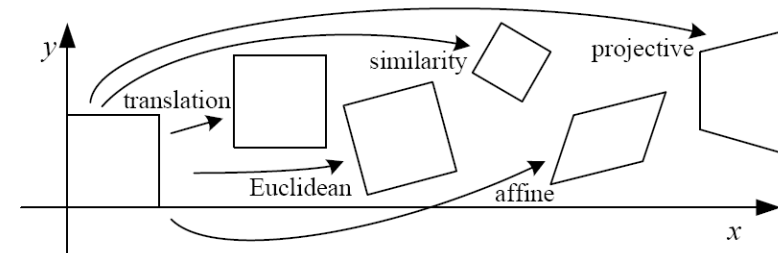
Classification of 2D transformations

Name	Matrix	# D.O.F.
translation	$\begin{bmatrix} \mathbf{I} & & \mathbf{t} \end{bmatrix}$	?
rigid (Euclidean)	$\begin{bmatrix} \mathbf{R} & & \mathbf{t} \end{bmatrix}$	?
similarity	$\begin{bmatrix} s\mathbf{R} & & \mathbf{t} \end{bmatrix}$	?
affine	$\begin{bmatrix} \mathbf{A} \end{bmatrix}$	?
projective	$\begin{bmatrix} \tilde{\mathbf{H}} \end{bmatrix}$	?

Classification of 2D transformations

Translation:

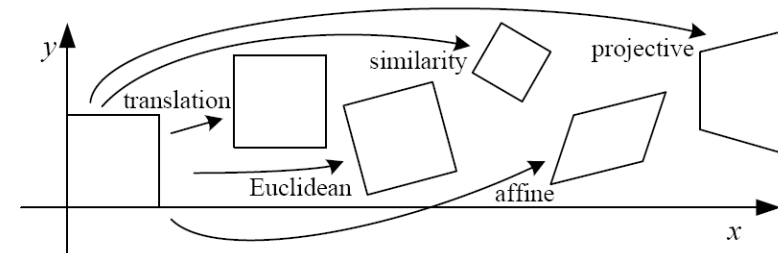
$$\begin{bmatrix} 1 & 0 & t_1 \\ 0 & 1 & t_2 \\ 0 & 0 & 1 \end{bmatrix}$$



Classification of 2D transformations

Euclidean (rigid):
rotation + translation

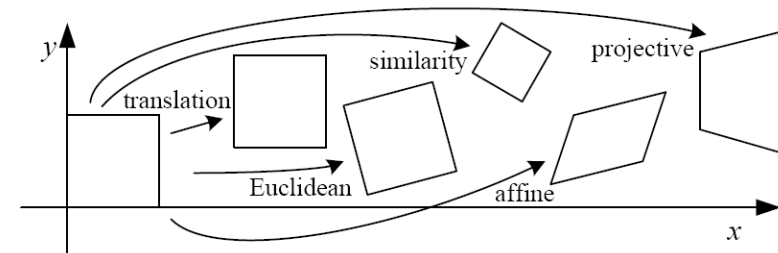
$$\begin{bmatrix} r_1 & r_2 & r_3 \\ r_4 & r_5 & r_6 \\ 0 & 0 & 1 \end{bmatrix}$$



Classification of 2D transformations

Euclidean (rigid):
rotation + translation

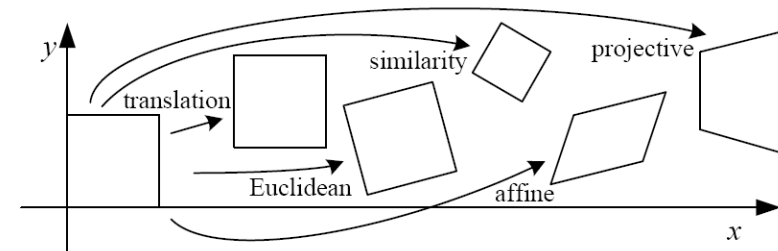
$$\begin{bmatrix} \cos \theta & -\sin \theta & r_3 \\ \sin \theta & \cos \theta & r_6 \\ 0 & 0 & 1 \end{bmatrix}$$



Classification of 2D transformations

Similarity:
uniform scaling + rotation
+ translation

$$\begin{bmatrix} r_1 & r_2 & r_3 \\ r_4 & r_5 & r_6 \\ 0 & 0 & 1 \end{bmatrix}$$



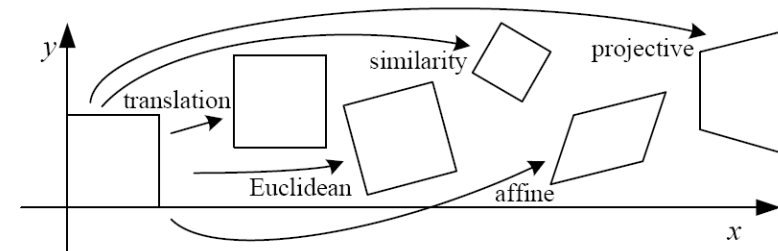
Classification of 2D transformations

multiply these four by scale s



Similarity:
uniform scaling + rotation
+ translation

$$\begin{bmatrix} \cos \theta & -\sin \theta & r_3 \\ \sin \theta & \cos \theta & r_6 \\ 0 & 0 & 1 \end{bmatrix}$$



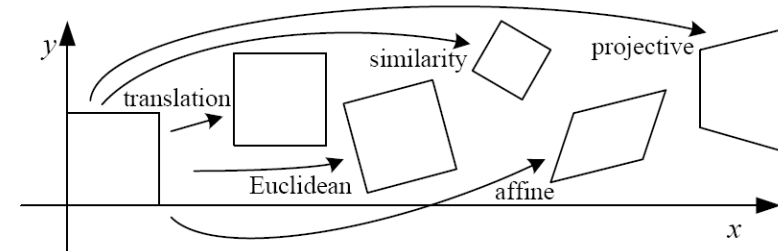
Classification of 2D transformations

Affine transform

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos(-\Phi) & -\sin(-\Phi) \\ \sin(-\Phi) & \cos(-\Phi) \end{bmatrix} \cdots \\ \cdots \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} \cos\Phi & -\sin\Phi \\ \sin\Phi & \cos\Phi \end{bmatrix}$$

Linear part can be decomposed



Classification of 2D transformations

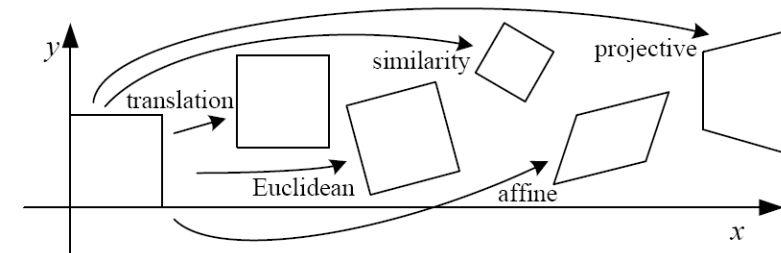
Affine transform

$$\mathbf{x}' = H_A \mathbf{x} = \begin{bmatrix} A & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix} \mathbf{x}$$

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos(-\Phi) & -\sin(-\Phi) \\ \sin(-\Phi) & \cos(-\Phi) \end{bmatrix} \cdots \\ \cdots \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} \cos\Phi & -\sin\Phi \\ \sin\Phi & \cos\Phi \end{bmatrix}$$

Linear part can be decomposed

$$A = R(\theta)R(-\Phi)D(\lambda_1, \lambda_2)R(\Phi)$$



Affine transformations

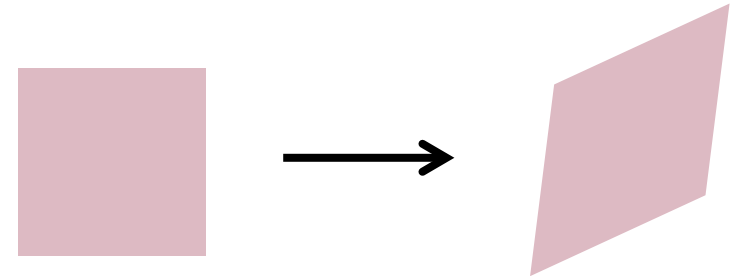
Affine transformations are combinations of

- arbitrary (4-DOF) linear transformations; and
- translations

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

Properties of affine transformations:

- origin does not necessarily map to origin
- lines map to lines
- parallel lines map to parallel lines
- ratios are preserved
- compositions of affine transforms are also affine transforms



Projective transformations

image point in
pixel coordinates

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

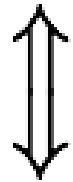
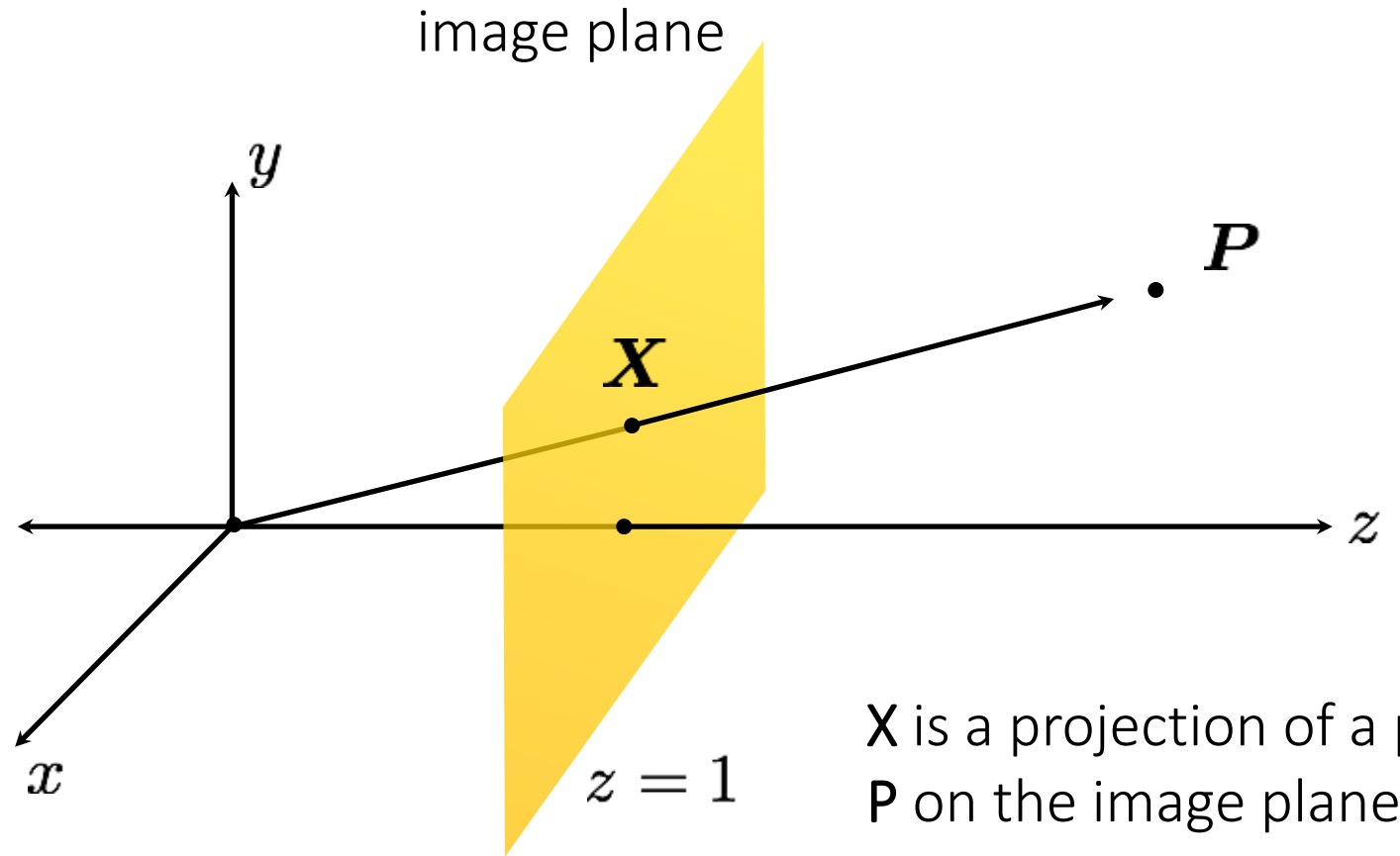


image point in
heterogeneous
coordinates

$$\mathbf{X} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$



$\mathbf{X}$ is a projection of a point
 $\mathbf{P}$ on the image plane

Projective transformations

Projective transformations are combinations of

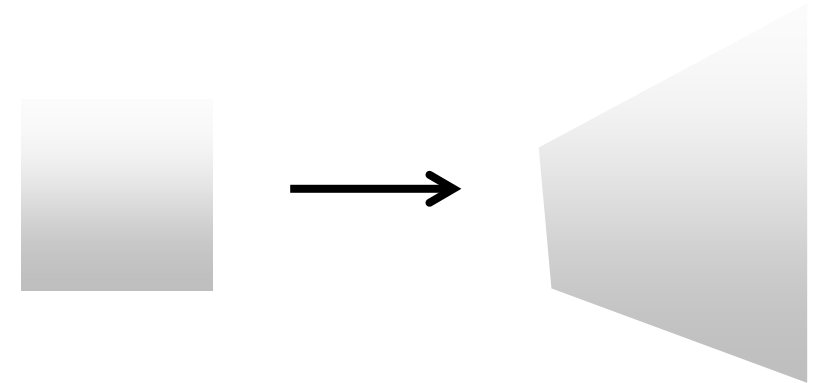
- affine transformations; and
- projective wraps

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

How many degrees of freedom?

Properties of projective transformations:

- origin does not necessarily map to origin
- lines map to lines
- parallel lines do not necessarily map to parallel lines
- ratios are not necessarily preserved
- compositions of projective transforms are also projective transforms

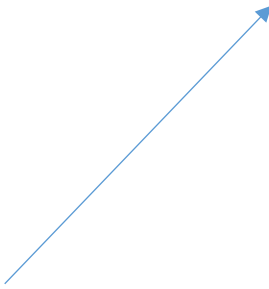


Projective transforms = 8Dof

$$k_{p2} \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} k_{p1} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$


$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \frac{k_{p1}}{k_{p2}} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$


$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \frac{k_{p1}}{k_{p2}} \begin{bmatrix} a_{11}/a_{33} & a_{12}/a_{33} & a_{13}/a_{33} \\ a_{21}/a_{33} & a_{22}/a_{33} & a_{23}/a_{33} \\ a_{31}/a_{33} & a_{32}/a_{33} & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$


$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = k \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Projective transformations

Projective transformations are combinations of

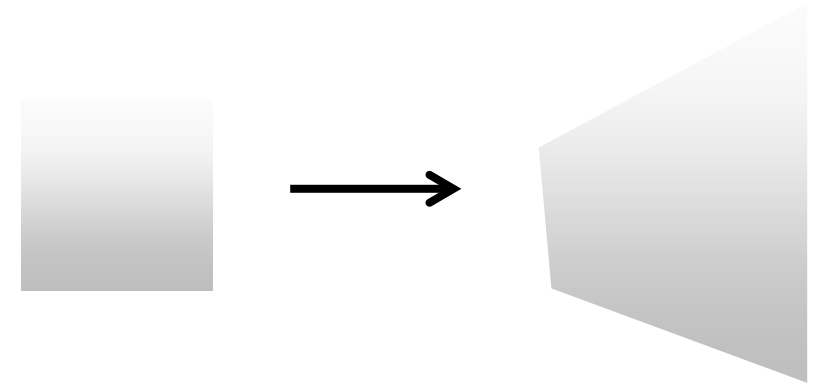
- affine transformations; and
- projective wraps

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

Properties of projective transformations:

- origin does not necessarily map to origin
- lines map to lines
- parallel lines do not necessarily map to parallel lines
- ratios are not necessarily preserved
- compositions of projective transforms are also projective transforms

8 DOF: vectors (and therefore matrices) are defined up to scale)



Classification of 2D transformations

Name	Matrix	# D.O.F.
translation	$\begin{bmatrix} \mathbf{I} & & \mathbf{t} \end{bmatrix}$	2
rigid (Euclidean)	$\begin{bmatrix} \mathbf{R} & & \mathbf{t} \end{bmatrix}$	3
similarity	$\begin{bmatrix} s\mathbf{R} & & \mathbf{t} \end{bmatrix}$	3
affine	$\begin{bmatrix} \mathbf{A} \end{bmatrix}$	6
projective	$\begin{bmatrix} \tilde{\mathbf{H}} \end{bmatrix}$	8

Properties

Group	Matrix	Distortion	Invariant properties
Projective 8 dof	$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$		Concurrency, collinearity, order of contact : intersection (1 pt contact); tangency (2 pt contact); inflections (3 pt contact with line); tangent discontinuities and cusps. cross ratio (ratio of ratio of lengths).
Affine 6 dof	$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Parallelism, ratio of areas, ratio of lengths on collinear or parallel lines (e.g. midpoints), linear combinations of vectors (e.g. centroids). The line at infinity, l_{∞} .
Similarity 4 dof	$\begin{bmatrix} sr_{11} & sr_{12} & t_x \\ sr_{21} & sr_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Ratio of lengths, angle. The circular points, I, J (see section 2.7.3).
Euclidean 3 dof	$\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Length, area