First-Order Logic & Higher-Order Logic, Knowledge-Based Agents

Material from Dr. Marie desJardin, Some material adopted from notes by Andreas Geyer-Schulz and Chuck Dyer

1

Bookkeeping

- HW4 out
- Designs will be graded this week – read the comments!
- Today's class:
- Review/finalize propositional logic
- Converting to CNF
- First-order logic
- Knowledge-based agents

2

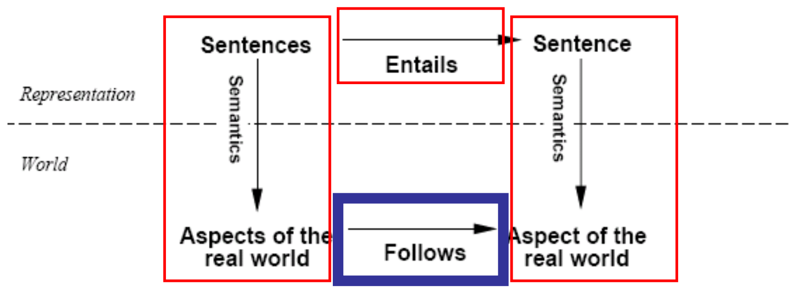
First-Order Logic

Chapter 8

Review

- Definitions:
  - Syntax, Semantics, Sentences, Propositions, Entails, Follows, Derives, Inference, Sound, Complete, Model, Satisfiable, Valid (or Tautology), etc.
- Syntactic Transformations:
  - E.g., $(A \Rightarrow B) \Leftrightarrow (\neg A \vee B)$
- Truth Tables
  - Negation, Conjunction, Disjunction, Implication, Equivalence (Biconditional)
  - Inference by Model Enumeration

Review: Schematic perspective



If KB is true in the real world,
*then any sentence α **entailed** by KB*
is also true in the real world.

For example: If I tell you (1) Sue is Mary's sister, and (2) Sue is Amy's mother, then it necessarily follows in the world that Mary is Amy's aunt, even though I told you nothing at all about aunts. This sort of reasoning pattern is what we hope to capture.

5

Review: Logic

- If a problem domain can be represented formally, then a decision maker can use **logical reasoning** to make rational decisions
- Many types of logic
  - Propositional Logic (Boolean logic)
  - First-Order Logic (aka first-order predicate calculus)
  - Non-Monotonic Logic
  - Markov Logic
- A logic includes:
  - syntax: what is a correctly-formed sentence?
  - semantics: what is the meaning of a sentence?
  - Inference procedure (reasoning, entailment): what sentence logically follows given knowledge?

6

Review: Propositional Logic

- A **symbol** in Propositional Logic (PL) is a symbolic variable whose value must be either True or False, and which stands for a natural language statement that could be either true or false
  - A = "Smith has chest pain"
  - B = "Smith is depressed"
  - C = "It is raining"

7

Review: Semantics

- An **interpretation** is a complete True / False assignment to all propositional symbols
  - Example symbols: P means "It is hot", Q means "It is humid", R means "It is raining"
  - There are 8 interpretations (TTT, ..., FFF)
- The semantics (meaning) of a sentence is the set of interpretations in which the sentence evaluates to True
- Example: the semantics of the sentence $P \vee Q$ is the set of 6 interpretations:
  - P=True, Q=True, R=True or False
  - P=True, Q=False, R=True or False
  - P=False, Q=True, R=True or False
- **A model of a set of sentences is an interpretation in which all the sentences are true**

8

Review: Knowledge Base (KB)

- A knowledge base, KB, is a set of sentences
  - Example KB:
  - HaveLecture ⇔ (TodayIsTuesday ∨ TodayIsThursday)
  - ¬HaveLecture
- It is equivalent to a single long sentence: the conjunction of all sentences
  - (HaveLecture ⇔ (TodayIsTuesday ∨ TodayIsThursday)) ∧ ¬HaveLecture
- **A model of a KB is an interpretation in which all sentences in KB are true**

9

Review: Entailment

- **Entailment** is the relation of a sentence β ***logically following*** from other sentences α (e.g., KB): **α ⊨ β**
- α ⊨ β if and only if, in every interpretation in which α is true, β is also true; i.e., whenever α is true, so is β
- Deduction theorem: **α ⊨ β if and only if α ⇒ β is valid** (always true)
- Proof by contradiction (refutation, *reductio ad absurdum*): **α ⊨ β if and only if α ∧ ¬β is unsatisfiable**
- There are 2^n interpretations to check, if KB has *n* symbols

10

So — how do we keep it from "Just making things up" ?

Is this inference correct?

How do you know?
How can you tell?

How can we **make correct** inferences?
How can we **avoid incorrect** inferences?

"Einstein Simplified: Cartoons on Science" by Sydney Harris, 1992, Rutgers University Press

11

So — how do we keep it from "Just making things up" ?

- All men are people;
- Half of all people are women;
- Therefore, half of all men are women.

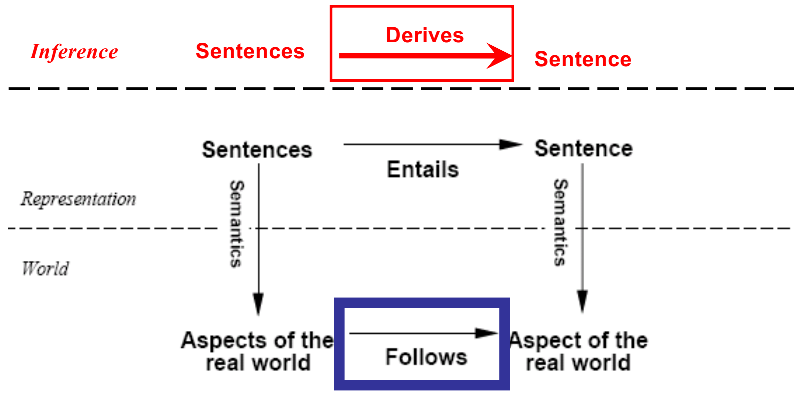
Is this inference correct?

How do you know?
How can you tell?

- Penguins are black and white;
- Some old TV shows are black and white;
- Therefore, some penguins are old TV shows.

12

Schematic perspective



If KB is true in the real world,

then any sentence α derived from KB

by a sound inference procedure

is also true in the real world.

13

Conjunctive Normal Form (CNF)

We'd like to prove: $\boxed{\begin{array}{l} KB \vDash \alpha \\ \text{(This is equivalent to } KB \wedge \neg \alpha \text{ is unsatisfiable.)} \end{array}}$

We first rewrite $KB \wedge \neg\alpha$ into **conjunctive normal form (CNF).**

$\downarrow$

A "conjunction of disjunctions"

literals

$$\underbrace{(A \vee \neg B)}_{\text{Clause}} \wedge \underbrace{(B \vee \neg C \vee \neg D)}_{\text{Clause}}$$

• Any KB can be converted into CNF.

14

Review: Equivalence & Implication

- Equivalence is a conjoined double implication
  - **$(X \Leftrightarrow Y) = [(X \Rightarrow Y) \wedge (Y \Rightarrow X)]$**
- Implication is (NOT antecedent OR consequent)
  - **$(X \Rightarrow Y) = (\neg X \vee Y)$**
  - How do we know this is true?
  - We can always use a truth table:

p	q	¬p	p ⇒ q	¬p V q
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

15

Review: de Morgan's rules

- How to bring ¬ inside parentheses
  - (1) Negate everything inside the parentheses
  - (2) Change operators to “the other operator”
- $\neg(X \wedge Y \wedge \ldots \wedge Z) = (\neg X \vee \neg Y \vee \ldots \vee \neg Z)$
- $\neg(X \vee Y \vee \ldots \vee Z) = (\neg X \wedge \neg Y \wedge \ldots \wedge \neg Z)$

16

Review: Boolean Distributive Laws

- Both of these laws are valid:
- AND distributes over OR
  - $X \wedge (Y \vee Z) = (X \wedge Y) \vee (X \wedge Z)$
  - $(W \vee X) \wedge (Y \vee Z) = (W \wedge Y) \vee (X \wedge Y) \vee (W \wedge Z) \vee (X \wedge Z)$
- OR distributes over AND
  - $X \vee (Y \wedge Z) = (X \vee Y) \wedge (X \vee Z)$
  - $(W \wedge X) \vee (Y \wedge Z) = (W \vee Y) \wedge (X \vee Y) \wedge (W \vee Z) \wedge (X \vee Z)$

17

Conjunctive Normal Form (CNF)

1. Replace all $\Leftrightarrow$ using iff/biconditional elimination
   - $\alpha \Leftrightarrow \beta \equiv (\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)$
2. Replace all $\Rightarrow$ using implication elimination
   - $\alpha \Rightarrow \beta \equiv \neg\alpha \vee \beta$
3. Move all negations inward using
   - double-negation elimination
     - $\neg(\neg\alpha) \equiv \alpha$
   - de Morgan's rule
     - $\neg(\alpha \vee \beta) \equiv \neg\alpha \wedge \neg\beta$
     - $\neg(\alpha \wedge \beta) \equiv \neg\alpha \vee \neg\beta$
4. Apply distributivity of $\vee$ over $\wedge$
   - $\alpha \wedge (\beta \vee \gamma) \equiv (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$ + 1 more

18

Convert Sentence into CNF

• $A \Leftrightarrow (B \vee C)$	starting sentence
• $(A \Rightarrow (B \vee C)) \wedge ((B \vee C) \Rightarrow A)$	iff/biconditional elimination
• $(\neg A \vee B \vee C) \wedge (\neg(B \vee C) \vee A)$	implication elimination
• $(\neg A \vee B \vee C) \wedge ((\neg B \wedge \neg C) \vee A)$	move negations inward
• $\underline{(\neg A \vee B \vee C)} \wedge (\neg B \vee A) \wedge (\neg C \vee A)$	distribute $\vee$ over $\wedge$

called a "**clause**"

19

Example: Conversion to CNF

Example: $B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})$

1. Eliminate $\Leftrightarrow$ by replacing $\alpha \Leftrightarrow \beta$ with $(\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)$.
   $= (B_{1,1} \Rightarrow (P_{1,2} \vee P_{2,1})) \wedge ((P_{1,2} \vee P_{2,1}) \Rightarrow B_{1,1})$

2. Eliminate $\Rightarrow$ by replacing $\alpha \Rightarrow \beta$ with $\neg\alpha \vee \beta$ and simplify.
   $= (\neg B_{1,1} \vee P_{1,2} \vee P_{2,1}) \wedge (\neg(P_{1,2} \vee P_{2,1}) \vee B_{1,1})$

3. Move $\neg$ inwards using de Morgan's rules and simplify.

$$\neg(\alpha \vee \beta) \equiv (\neg\alpha \wedge \neg\beta),\ \neg(\alpha \wedge \beta) \equiv (\neg\alpha \vee \neg\beta)$$

   $= (\neg B_{1,1} \vee P_{1,2} \vee P_{2,1}) \wedge ((\neg P_{1,2} \wedge \neg P_{2,1}) \vee B_{1,1})$

4. Apply distributive law ($\wedge$ over $\vee$) and simplify.
   $= (\neg B_{1,1} \vee P_{1,2} \vee P_{2,1}) \wedge (\neg P_{1,2} \vee B_{1,1}) \wedge (\neg P_{2,1} \vee B_{1,1})$

20

Example: Conversion to CNF

- Example: B1,1 $\Leftrightarrow$ (P1,2 $\vee$ P2,1)
- From the previous slide we had:
  - = ($\neg$B1,1 $\vee$ P1,2 $\vee$ P2,1) $\wedge$ ($\neg$P1,2 $\vee$ B1,1) $\wedge$ ($\neg$P2,1 $\vee$ B1,1)

5. KB is the conjunction of all of its sentences (all are true),
   - so write each clause (disjunct) as a sentence in KB:

- KB = …
  ($\neg$B1,1 $\vee$ P1,2 $\vee$ P2,1)
  ($\neg$P1,2 $\vee$ B1,1)
  ($\neg$P2,1 $\vee$ B1,1)
  …

Often, Won't Write "$\wedge$"
(we know it is there)

- Can do this in Propositional Logic, but often we want to use First Order Logic

21

First-Order Logic

- First-order logic (FOL) models the world in terms of
  - **Objects**, which are things with individual identities
  - **Properties** of objects that distinguish them from other objects
  - **Relations** that hold among sets of objects
  - **Functions**, which are a subset of relations where there is only one "value" for any given "input"
- Examples:
  - Objects: students, lectures, companies, cars …
  - Relations: brother-of, bigger-than, outside, part-of, has-color, occurs-after, owns, visits, precedes, …
  - Properties: blue, oval, even, large, …
  - Functions: father-of, best-friend, second-half, one-more-than …

22

User Provides

- **Constant symbols**, which represent individuals in the world
  - Mary
  - 3
  - Green
- **Function symbols**, which map individuals to individuals
  - father-of(Mary) = John
  - color-of(Sky) = Blue
- **Predicate symbols**, which map individuals to truth values
  - greater(5,3)
  - green(Grass)
  - color(Grass, Green)

23

FOL Provides

- **Variable symbols**
  - E.g., x, y, foo
- **Connectives**
  - Same as in PL: not ($\neg$), and ($\wedge$), or ($\vee$), implies ($\Rightarrow$), if and only if (biconditional $\leftrightarrow$)
- **Quantifiers**
  - Universal $\forall$x or (Ax)
  - Existential $\exists$x or (Ex)

24

Sentences: Terms and Atoms

- A **term** (denoting a real-world individual) is:
  - A constant symbol: *John*, or
  - A variable symbol: *x*, or
  - An n-place function of n terms

    x and $f(x_1, ..., x_n)$ are terms, where each x_i is a term

    is-a(John, Professor)
  - A term with no variables is a **ground term**.
- An **atomic sentence** is an n-place predicate of n terms
  - Has a truth value (*t* or *f*)

25

First-Order Logic (FOL

- Propositional logic assumes the world contains facts.
- First-order logic (like natural language) assumes the world contains
  - **Objects:** people, houses, numbers, colors, baseball games, wars, …
  - **Functions**: father of, best friend, one more than, plus, …
    - Function arguments are objects; function returns an object
  - Objects generally correspond to English NOUNS
  - **Predicates/Relations/Properties:** red, round, prime, brother of, bigger than, part of, between…
    - Predicate arguments are objects; predicate returns a truth value
  - Predicates generally correspond to English VERBS
    - **First argument is generally the subject, the second the object**
    - Hit(Bill, Ball) usually means "Bill hit the ball."
    - Likes(Bill, IceCream) usually means "Bill likes IceCream."
    - Verb(Noun1, Noun2) usually means "Noun1 verb noun2."

26

Syntax of FOL: Atomic Sentences

- Atomic sentences in logic state facts that are true or false.
- Properties and m-ary relations do just that:
  - LargerThan(2, 3) is false.
  - BrotherOf(Mary, Jane) is false.
  - Married(Father(Richard), Mother(John)) could be true or false.
- Note: Functions refer to objects, do not state facts, and form no sentence:
  - Brother(Pete) refers to John (his brother) and is neither true nor false.
  - Plus(2, 3) refers to the number 5 and is neither true nor false.
- BrotherOf(Pete, Brother(Pete)) is True.

Binary relation is a truth value.

Function refers to John, an object in the world, i.e., John is Pete's brother.

27

Syntax of FOL: Variables

- **Variables** range over objects in the world.
- A **variable** is like a **term** because it represents an object.
- A **variable** may be used wherever a **term** may be used.
  - **Variables** may be arguments to functions and predicates.
- (A **term with NO variables** is called a **ground term**.)
- (A **variable not bound by a quantifier** is called **free**.

28

Syntax of FOL: Basic syntax elements are symbols

- Constant Symbols (correspond to English nouns)
  - Stand for objects in the world. E.g., KingJohn, 2, France, …
- Predicate Symbols (correspond to English verbs)
  - Stand for relations (maps a tuple of objects to a truth-value)
    - E.g., Brother(Richard, John), greater_than(3,2), …
- Function Symbols (correspond to English nouns)
  - Stand for functions (maps a tuple of objects to an object)
    - E.g., Sqrt(3), LeftLegOf(John), …
- Model (world) = set of domain objects, relations, functions
- Interpretation maps symbols onto the model (world)
  - Very many interpretations are possible for each KB and world!
  - Job of the KB is to rule out models inconsistent with our knowledge.

29

Syntax of FOL: Basic elements

• Constants	KingJohn, 2, UMBC,…	
• Predicates	BrotherOf, >,…	(return true or false)
• Functions	Sqrt, LeftLegOf,…	(return some object)
• Variables	x, y, a, b,…	
• Quantifiers	$\forall$, $\exists$	
• Connectives	$\neg$, $\wedge$, $\vee$, $\Rightarrow$, $\Leftrightarrow$ (standard)	
• Equality	= (but causes difficulties….)	

30

Sentences: Terms and Atoms

- A **complex sentence** is formed from atomic sentences connected by the same logical connectives as in propositional logic:
  $\neg$P, P$\vee$Q, P$\wedge$Q, P$\Rightarrow$Q, P$\leftrightarrow$Q where P and Q are sentences

has-a(x, Bachelors) $\wedge$ *is-a(x, human)*

does NOT SAY everyone with a bachelors' is human

has-a(John, Bachelors) $\wedge$ *is-a(John, human)*

has-a(Mary, Bachelors)

31

Quantifiers

- **Universal quantification**
  - $\forall$x P(x) means that P holds for **all** values of *x* in its domain
  - States universal truths
  - E.g.: $\forall$*x dolphin(x)* $\Rightarrow$ *mammal(x)*
- **Existential quantification**
  - $\exists$x P(x) means that P holds for **some** value of x in the domain associated with that variable
  - Makes a statement about some object without naming it
  - E.g., $\exists$*x mammal(x)* $\wedge$ *lays-eggs(x)*

32

Sentences: Quantification

- **Quantified sentences** adds quantifiers ∀ and ∃

  $\forall x\ has\text{-}a(x, Bachelors) \Rightarrow is\text{-}a(x, human)$

  $\exists x\ has\text{-}a(x, Bachelors)$

  $\forall x\ \exists y\ Loves(x, y)$

Everyone who has a bachelors' is human.

There exists some who has a bachelors'.

Everybody loves somebody.

33

Sentences: Well-Formedness

- A **well-formed formula (wff)** is a sentence containing no "free" variables. That is, all variables are "bound" by universal or existential quantifiers.

- $(\forall x)P(x,y)$ has x bound as a universally quantified variable, but y is free.

34

Quantifiers: Uses of $\forall$

- Universal quantifiers **often** used with “implies” to form “rules”:

  $(\forall x)$ student(x) $\Rightarrow$ smart(x)

  “All students are smart”

- Universal quantification **rarely*** used to make blanket statements about every individual in the world:

  $(\forall x)$student(x)$\wedge$smart(x)

  “Everyone in the world is a student and is smart”

*Deliberately, anyway

35

Quantifiers: Uses of $\exists$

- Existential quantifiers are **usually** used with “and” to specify a list of properties about an individual:
  - $(\exists x)$ student(x) $\wedge$ smart(x)
  - “There is a student who is smart”
- A common mistake is to represent this English sentence as the FOL sentence:
  - $(\exists x)$ student(x) $\Rightarrow$ smart(x)
  - But what happens when there is a person who is not a student?

36

Translation with Quantifiers

- Universal statements typically use **implications**
  - All S(x) is P(x):
    - $\forall x(S(x) \Rightarrow P(x))$
  - No S(x) is P(x):
    - $\forall x(S(x) \Rightarrow \neg P(x))$
- Existential statements typically use **conjunctions**
  - Some S(x) is P(x):
    - $\exists x (S(x) \wedge P(x))$
  - Some S(x) is not P(x):
    - $\exists x (S(x) \wedge \neg P(x))$

37

Quantifier Scope

- Switching the order of universal quantifiers does not change the meaning:
  - $(\forall x)(\forall y)P(x,y) \leftrightarrow (\forall y)(\forall x) P(x,y)$
- Similarly, you can switch the order of existential quantifiers:
  - $(\exists x)(\exists y)P(x,y) \leftrightarrow (\exists y)(\exists x) P(x,y)$
- Switching the order of universals and existentials does change meaning:
  - Everyone likes someone: $(\forall x)(\exists y)$ likes(x,y)
  - Someone is liked by everyone: $(\exists y)(\forall x)$ likes(x,y)

38

Connections between All and Exists

- We can relate sentences involving $\forall$ and $\exists$ using De Morgan's laws:
  - $(\forall x)\ \neg P(x) \leftrightarrow \neg(\exists x)\ P(x)$
  - $\neg(\forall x)\ P \leftrightarrow (\exists x)\ \neg P(x)$
  - $(\forall x)\ P(x) \leftrightarrow \neg(\exists x)\ \neg P(x)$
  - $(\exists x)\ P(x) \leftrightarrow \neg(\forall x)\ \neg P(x)$

39

Quantified Inference Rules

- Universal instantiation
  - $\forall x\ P(x)\ \therefore\ P(A)$
- Universal generalization
  - $P(A) \wedge P(B) \ldots\ \therefore\ \forall x\ P(x)$
- Existential instantiation
  - $\exists x\ P(x)\ \therefore P(F)$ ← **skolem constant F**
- Existential generalization
  - $P(A)\ \therefore\ \exists x\ P(x)$

40

Universal Instantiation (a.k.a. Universal Elimination)

- If $(\forall x)\ P(x)$ is true, then $P(C)$ is true, where C is any constant in the domain of x
- Example:
  - $(\forall x)$ eats(Ziggy, x) $\Rightarrow$ eats(Ziggy, IceCream)
- The variable symbol can be replaced by any ground term, i.e., any constant symbol or function symbol applied to ground terms only

41

Existential Instantiation (a.k.a. Existential Elimination)

- Variable is replaced by a **brand-new constant**
  - I.e., not occurring in the KB
- From $(\exists x)\ P(x)$ infer $P(c)$
  - Example:
    - $(\exists x)$ eats(Ziggy, x) $\Rightarrow$ eats(Ziggy, Stuff)
  - "Skolemization" – create a new term that instantiates existence
- *Stuff* is a **skolem constant**
- Easier than manipulating the existential quantifier

42

Existential Generalization (a.k.a. Existential Introduction)

- If P(c) is true, then $(\exists x)$ P(x) is inferred.
- Example
  - eats(Ziggy, IceCream) $\Rightarrow$ $(\exists x)$ eats(Ziggy, x)
- All instances of the given constant symbol are replaced by the new variable symbol
- Note that the variable symbol cannot already exist anywhere in the expression

43

Translating English to FOL

- **Every gardener likes the sun.**
  - $\forall x$ gardener(x) $\Rightarrow$ likes(x,Sun)
- **You can fool some of the people all of the time.**
  - $\exists x\ \forall t$ person(x) $\wedge$time(t) $\Rightarrow$ can-fool(x,t)
- **You can fool all of the people some of the time.**
  - $\forall x\ \exists t$ (person(x) $\Rightarrow$ time(t) $\wedge$can-fool(x,t))
  - $\forall x$ (person(x) $\Rightarrow$ $\exists t$ (time(t) $\wedge$can-fool(x,t))

Equivalent

- **All purple mushrooms are poisonous.**
  - $\forall x$ (mushroom(x) $\wedge$ purple(x)) $\Rightarrow$ poisonous(x)

44

Translating English to FOL

- **No purple mushroom is poisonous.**
  - $\neg\exists x$ purple(x) $\wedge$ mushroom(x) $\wedge$ poisonous(x)
  - $\forall x$ (mushroom(x) $\wedge$ purple(x)) $\Rightarrow \neg$poisonous(x)

Equivalent

- **There are exactly two purple mushrooms.**
  - $\exists x\, \exists y$ mushroom(x) $\wedge$ purple(x) $\wedge$ mushroom(y) $\wedge$ purple(y) ^ $\neg$(x=y) $\wedge$ $\forall z$ (mushroom(z) $\wedge$ purple(z)) $\Rightarrow$ ((x=z) $\vee$ (y=z))
- **Mary is not tall.**
  - $\neg$tall(Mary)
- **X is above Y iff X is on directly on top of Y or there is a pile of one or more other objects directly on top of one another starting with X and ending with Y.**
  - $\forall x\, \forall y$ above(x,y) $\Leftrightarrow$ (on(x,y) $\vee$ $\exists z$ (on(x,z) $\wedge$ above(z,y)))

45

Semantics of FOL

- **Domain M:** the set of all objects in the world (of interest)
- **Interpretation I:** includes
  - Assign each constant to an object in M
  - Define each function of n arguments as a mapping M^n => M
  - Define each predicate of n arguments as a mapping M^n => {T, F}
  - Therefore, every ground predicate with any instantiation will have a truth value
  - In general there is an infinite number of interpretations because |M| is infinite
- **Define logical connectives:** ~, ^, v, =>, <=> as in PL
- **Define semantics of ($\forall$x) and ($\exists$x)**
  - ($\forall$x) P(x) is true iff P(x) is true under all interpretations
  - ($\exists$x) P(x) is true iff P(x) is true under some interpretation

46

Terminology

- **Model**: an interpretation of a set of sentences such that every sentence is True
- A sentence is
  - **Satisfiable** if it is true under some interpretation
  - **Valid** if it is true under all possible interpretations
  - **Inconsistent** if there does not exist any interpretation under which the sentence is true
- **Logical consequence**: S ⊨ X if all models of S are also models of X

47

Axioms, Definitions and Theorems

- **Axioms** are facts and rules that attempt to capture all of the (important) facts and concepts about a domain; axioms can be used to prove **theorems**
  - Mathematicians don't want any unnecessary (dependent) axioms –ones that can be derived from other axioms
  - Dependent axioms can make reasoning faster, however
  - Choosing a good set of axioms for a domain is a kind of design problem
- A definition of a predicate is of the form "p(X) ↔ …" and can be decomposed into two parts
  - **Necessary** description: "p(x) ⇒ …"
  - **Sufficient** description "… ⇒ p(x)"
  - Some concepts don't have complete definitions (e.g., person(x))

48

Necessary and Sufficient

- p is **necessary** for q
  - ¬p ⇒ ¬q (“no p, no q!”)
- p is **sufficient** for q
  - p ⇒ q (“p is all we need to know!”)
- Note that ¬p ⇒ ¬q is equivalent to q ⇒ p
- So if p is necessary and sufficient for q, then p iff q.

49

More on Definitions

- Examples: define father(x, y) by parent(x, y) and male(x)
  - parent(x, y) is a **necessary (but not sufficient**) description of father(x, y)
    - father(x, y) ⇒ parent(x, y)
  - parent(x, y) ^ male(x) ^ age(x, 35) is a **sufficient (but not necessary**) description of father(x, y):
    - father(x, y) ← parent(x, y) ^ male(x) ^ age(x, 35)
  - parent(x, y) ^ male(x) is a **necessary and sufficient** description of father(x, y)
    - parent(x, y) ^ male(x) ↔ father(x, y)

50

Converting FOL to CNF

- Eliminate biconditionals and implications
- Move ¬ inwards
- **Standardize variables apart** by renaming them: each quantifier should use a different variable
- **Skolemize:** each existential variable is replaced by a Skolem constant or Skolem function of the enclosing universally quantified variables.
  - For instance, ∃x Rich(x) becomes Rich(G1) where G1 is a new Skolem constant
  - "Everyone has a heart" [∀x Person(x) ⇒ ∃y Heart(y)∧Has(x,y)] becomes
    ∀x Person(x) ⇒ Heart(H(x)) ∧ Has(x, H(x)), where H is a new symbol (Skolem function)
- **Drop universal quantifiers**
  - For instance, ∀ x Person(x) becomes Person(x).
- Distribute ∧ over ∨

51

Summary: First Order Logic (FOL)

- Uses the same logical symbols as Propositional Logic (PL)
- Adds: variables, quantification, predicates and functions
  - Names of terms: **constants**, **variables**, **predicates**, **functions**
- Existential and universal quantifiers can be used to create rules
- Need to be able to translate English to and from FOL
- Some extensions…

52

Higher-Order Logic

- FOL only allows to quantify over variables, and variables can only range over objects.
- HOL allows us to quantify over relations
- Example: (quantify over functions)
  - “two functions are equal iff they produce the same value for all arguments”
  - $\forall f\ \forall g\ (f = g) \leftrightarrow (\forall x\ f(x) = g(x))$
- Example: (quantify over predicates)
  - $\forall r$ transitive(r) $\Rightarrow (\forall xyz)\ r(x,y) \wedge r(y,z) \Rightarrow r(x,z))$
- More expressive, but undecidable.

53

Expressing Uniqueness

- Sometimes we want to say that there is a single, unique object that satisfies a certain condition
- “There exists a unique x such that king(x) is true”
  - $\exists x\ king(x) \wedge \forall y\ (king(y) \Rightarrow x=y)$
  - $\exists x\ king(x) \wedge \neg\exists y\ (king(y) \wedge x\neq y)$
  - $\exists!\ x\ king(x)$
- “Every country has exactly one ruler”
  - $\forall c\ country(c) \Rightarrow \exists!\ r\ ruler(c,r)$
- Iota operator: “ι x P(x)” means “the unique x such that p(x) is true”
  - “The unique ruler of Freedonia is dead”
  - dead(ι x ruler(freedonia,x))

54

Notational differences

- Different symbols for and, or, not, implies, ...
  - $\forall$ $\exists$ $\Rightarrow$ $\Leftrightarrow$ $\wedge$ $\vee$ $\neg$ $\bullet$ $\supset$
  - p v (q ^ r)
  - p + (q * r)
  - etc
- Prolog
  - cat(X) :- furry(X), meows (X), has(X, claws)
- Lispy notations

```
(forall ?x (implies (and (furry ?x)
                         (meows ?x)
                         (has ?x claws))
                    (cat ?x)))
```

55

Exercise: FOL translation

1. Everything is bitter or sweet.
2. Either everything is bitter or everything is sweet.
3. There is somebody who is loved by everyone.
4. Nobody is loved by no one.
5. If someone is noisy, everybody is annoyed
6. Frogs are green.
7. Frogs are not green.
8. No frog is green.
9. Some frogs are not green.
10. A mechanic likes Bob.
11. A mechanic likes herself.
12. Every mechanic likes Bob.
13. Some mechanic likes every nurse.
14. There is a mechanic who is liked by every nurse.

Exercises: disi.unitn.it/~bernardi/Courses/LSNL/Slides/fl1.pdf

56

Exercise: FOL translation

1. $\forall x\ (bitter(x) \lor sweet(x))$
2. $\forall x\ (bitter(x)) \lor \forall x\ (sweet(x))$
3. $\exists x\ \forall y\ (loves(y,x))$
4. $\neg\exists x\ \neg\exists y\ (loves(y,x))$
5. $\exists x\ (noisy(x)) \Rightarrow \forall y(annoyed(y))$
6. $\forall x\ (frog(x) \Rightarrow green(x))$
7. $\forall x\ (frog(x) \Rightarrow \neg green(x))$
8. $\neg\exists x\ (frog(x) \land green(x))$
9. $\exists x\ (frog(x) \land \neg green(x))$
10. $\exists x\ (mech.(x) \land likes(x, Bob))$
11. $\exists x\ (mech.(x) \land likes(x, x))$
12. $\forall x\ (mech.(x) \Rightarrow likes(x, Bob))$
13. $\exists x\ \forall y\ (mech(x) \land nurse(y) \Rightarrow likes(x, y))$
14. $\exists x(mech(x) \land \forall y(nurse(y) \Rightarrow likes(y, x))$

Exercises: disi.unitn.it/~bernardi/Courses/LSNL/Slides/fl1.pdf

57