Quadric Error Metrics

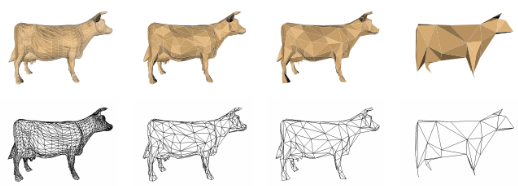


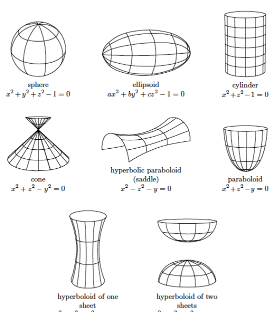
Figure 2.7: Fixed set of levels of detail for a cow model.

To do

- Continue to work on ray programming assignment
- Continue to think about final project
- Next Monday: Dr. Lee Butler's guest lecture

Background: quadrics

- A quadrics are all surfaces that can be expressed as a second degree polynomial in x, y, and z.



Quadric surfaces

- General implicit form

$$ax^2+2bxy+2cxz+2dxw+ey^2+2fyz+2gyw+hz^2+2izw+jw^2=0.$$

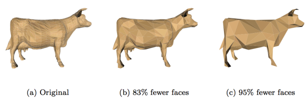
- Matrix form
  - Setting w to 1, this provides the ability to position the quadrics in space

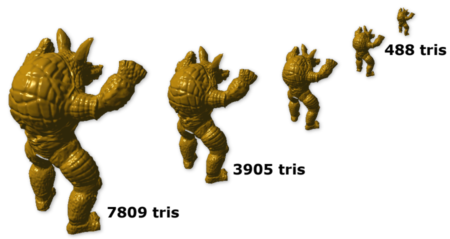
$$\mathrm{x}^tQ\mathrm{x}=0,$$

$$\text{where } \mathrm{x}=\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix},\ \mathrm{x}^t=\begin{bmatrix} x & y & z & w \end{bmatrix}, \text{ and } Q=\begin{bmatrix} a & b & c & d \\ b & e & f & g \\ c & f & h & i \\ d & g & i & j \end{bmatrix}$$

Surface Simplification: Goals

- Efficiency (70000 to 100 faces in 15s in 1997)
- High quality, feature preserving (primary appearance emphasized rather than topology)
- Generally, non-manifold models, collapse disjoint regions





Background: Manifold

- Mathematical term of a surface, all of those points have a neighborhood which is topologically equivalent to a disk.
- A manifold with boundary is a surface all of whose points have either a disk or a half-disk.
- A polygonal surface is a manifold (with boundary) if every edge has exactly two incident faces, and the neighborhood of every vertex consists of a closed loop of faces (or a single fan of faces on the boundary)

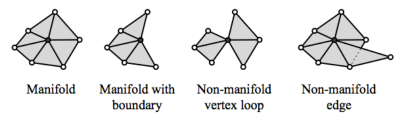
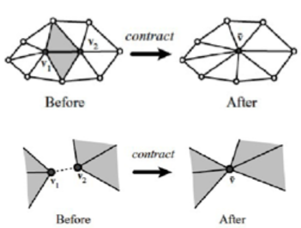


Figure 2.1: Neighborhoods of a given vertex (in black). On the left, two manifold neighborhoods. On the right, two non-manifold neighborhoods.

Overview and Resources

- Garland and Heckbert SIGGRAPH 97 paper
  - Greedy decimation algorithm
  - Pair collapse (allow edge + non-edge collapse)
- Evaluate potential collapse
- Determine optimal new vertex locations



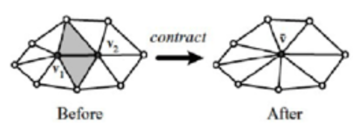
- Garland website, implementation notes in his thesis

http://mgarland.org/research/quadrics.html
http://mgarland.org/research/thesis.html

- Notes in this and the previous lecture

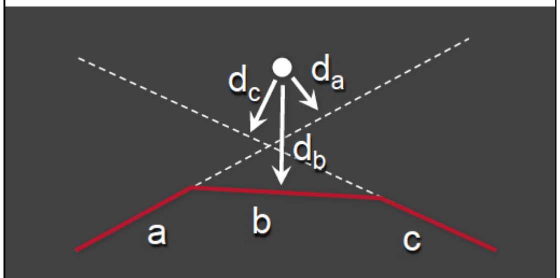
Algorithm Outline

- Restrict process to a set of *valid pairs*:
  - $(\mathbf{v}_i, \mathbf{v}_j)$ is an edge, or
  - $||\mathbf{v}_i - \mathbf{v}_j|| < t$, a threshold
    - $t = 0$ restricts to edge contraction
    - $t \gg 0$ can connect distant regions or yield $O(n^2)$ pairs
- Iteratively remove *best* pair and update valid pairs list:
  - Each vertex has a set with the pairs it belongs to:
    - $\mathbf{v}_i \rightarrow \text{Pairs}(\mathbf{v}_i)$
    - $(\mathbf{v}_i, \mathbf{v}_j) \rightarrow \overline{\mathbf{v}} \Rightarrow \text{Pairs}(\overline{\mathbf{v}}) = \text{Pairs}(\mathbf{v}_i) \cup \text{Pairs}(\mathbf{v}_j)$
- But how to choose *best* pair?



Choose best pair

- Based on point-to-plane distance
- Better quality than point-to-point



Background: Computing Planes

- Each triangle in mesh has associated plane

$$ax+by+cz+d=0$$

- For a triangle, find its (normalized) normal using cross products

$$\vec{n}=\frac{AB\times AC}{|AB\times AC|} \qquad \vec{n}\cdot\vec{v}-\vec{A}\cdot\vec{v}=0$$

- Plane equation:

$$\vec{n}=\begin{pmatrix}a\\b\\c\end{pmatrix} \qquad d=-\vec{A}\cdot\vec{v}$$

Quadric Error Metrics

- Sum of squared distances from vertex to planes

$$\Delta_{\mathbf{v}}=\sum_{\mathbf{p}} Dist(\mathbf{v},\mathbf{p})^2$$

$$\mathbf{v}=\begin{pmatrix}x\\y\\z\\1\end{pmatrix},\quad \mathbf{p}=\begin{pmatrix}a\\b\\c\\d\end{pmatrix}$$

$$Dist(\mathbf{v},\mathbf{p})=ax+by+cz+d=\mathbf{p}^T\mathbf{v}$$

$$\Delta=\sum_{\mathbf{p}}(\mathbf{p}^T\mathbf{v})^2 =\sum_{\mathbf{p}}\mathbf{v}^T\mathbf{p}\mathbf{p}^T\mathbf{v} =\mathbf{v}^T\left(\sum_{\mathbf{p}}\mathbf{p}\mathbf{p}^T\right)\mathbf{v} =\mathbf{v}^T\mathbf{Q}\mathbf{v}$$

- Common mathematical trick:

Quadratic form = symmetric matrix Q multiplied twice by a vector.

Quadric Error Metrics

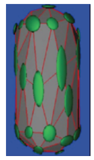
- A 4x4 symmetric matrix
- Storage efficient: 10 floating point number per vertex
- Initially, error is 0 for all vertices

$$\mathbf{v}^T\mathbf{Q}\mathbf{v}=[x\ \ y\ \ z\ \ 1]\begin{bmatrix}q_{11} & q_{12} & q_{13} & q_{14}\\ q_{12} & q_{22} & q_{23} & q_{24}\\ q_{13} & q_{23} & q_{33} & q_{34}\\ q_{14} & q_{24} & q_{34} & q_{44}\end{bmatrix}\begin{bmatrix}x\\y\\z\\1\end{bmatrix}$$

- 2^{nd} degree polynomial in x, y, and z
- Level surface ($\mathbf{v}^t Q\mathbf{v}$ = k) is a quadric surface
  - Ellipsoid, parabolic surface, hyperboloid, plane etc.

$$\mathbf{v}^T\mathbf{Q}\mathbf{v}=q_{11}x^2+2q_{12}xy+2q_{13}xz+2q_{14}x +q_{22}y^2+2q_{23}yz+2q_{24} +q_{33}z^2+2q_{34}z+q_{44}$$

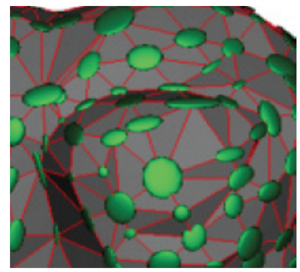
But What Are These Quadrics Really Doing?



- Almost always ellipsoids
  - When Q is positive definite
- Characterize error at vertex
  - Vertex at center of each ellipsoid
  - Move it anywhere on ellipsoid with constant error
- Capture local shape of surface
  - Stretch in least curved direction

Quadric Visualization

- Ellipsoids: iso-error surfaces
- Smaller ellipsoid = greater error for a given motion
- Lower error for motion parallel to surface
- Lower error in flat regions than at corners
- Elongated in "cylindrical" regions



Using Quadrics

- Approximate error of edge collapses
  - Each vertex v has associated quadric Q
  - Error of collapsing v_1 and v_2 to v' is $v'^t Q_1 v' + v'^t Q_2 v'$
  - Quadric for new vertex v' is Q' = Q1 + Q2

- Find optimal location v' after collapse

$$\mathbf{Q'} = \begin{bmatrix} q_{11} & q_{12} & q_{13} & q_{14} \\ q_{12} & q_{22} & q_{23} & q_{24} \\ q_{13} & q_{23} & q_{33} & q_{34} \\ q_{14} & q_{24} & q_{34} & q_{44} \end{bmatrix}$$

$$\min_{\mathbf{v'}} \mathbf{v'}^T \mathbf{Q'} \mathbf{v'}: \quad \frac{\partial}{\partial x} = \frac{\partial}{\partial y} = \frac{\partial}{\partial z} = 0$$

$$\begin{bmatrix} q_{11} & q_{12} & q_{13} & q_{14} \\ q_{12} & q_{22} & q_{23} & q_{24} \\ q_{13} & q_{23} & q_{33} & q_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{v'} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{v'} = \begin{bmatrix} q_{11} & q_{12} & q_{13} & q_{14} \\ q_{12} & q_{22} & q_{23} & q_{24} \\ q_{13} & q_{23} & q_{33} & q_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Algorithm Summary

- Compute the Q matrices for all the initial vertices
- Select all valid pairs
- Computer the optimal contraction target v for each valid pair. The error $v^t(Q_1+Q_2)$ v′ of this target vertex becomes the **cost** of contracting that pair.
- Place all pairs in a heap keyed on cost with minimum cost pair on the top
- Iteratively remove the least cost pair, contract this pair, and update the costs of all valid pairs of interest

- Compute Q_i for all vertices $\mathbf{v}_i$
- Determine valid pairs
- Compute optimal contraction target and associated quadric error for each pair
- Place all pairs in a heap, ordered by smallest error
- Repeat
  - Get least error pair $(\mathbf{v}_i, \mathbf{v}_j)$ from heap
  - Contract pair (move edges to $\mathbf{v}$, remove degenerate planes)
  - Update cost for all pairs involving $\mathbf{v}_i$ and $\mathbf{v}_j$
- Until done.

Results

Original Quadrics 1000 tris 100 tris

Original Quadrics

250 tris 250 tris, edge collapses only

Additional Details

- Preserving boundaries / discontinuities (weight quadrics by appropriate penalty factors)
- Preventing mesh inversion (flipping of orientation): compare normal of neighboring faces, before after
- Has been modified for many other applications
  - E.g., in silhouettes, want to make sure volume always increases, never decreases
  - Take color and texture into account (follow up paper)

Implementation Tips

- Incremental, test, debug, simple cases
- Find good data structure for heap etc.
- May help to visualize error quadrics if possible
- Challenging, but hopefully rewarding assignment