Assignment 8

Problem 1: Cubic Equations

In this problem, you will use the Math package *Maple*[1] to derive a closed-form solution to the cubic equation

$$f(x) = ax^3 + bx^2 + cx + d = 0, \tag{1}$$

where a, b, c, d are real constants, and $a \neq 0$. Using *Maple*, complete each of the following tasks:

a) Login to any workstation that runs *Maple*, and start a transcript of your terminal session session (*e.g.* use the Unix script command). Enter *Maple* by typing "maple". Define the polynomial f.

b) Apply the substitution $x = y - (b/(3a))$ to express Equation 1 in standard form $g(y) = y^3 + qy - r = 0$. What are the coefficients q and r? *Hint:* Type "help(subs);" and "help(simplify);".

c) Apply the substitution $y = u + v$ to $g(y)$. What expression do you get?

d) Letting $v = -q/(3u)$, simplify the resulting expression as a 6th-degree polynomial in u, which we shall call $h(u)$. What is $h(u)$?

e) Express $h(u)$ as a quadratic equation in u^3. Solve this quadratic equation to get a closed-form expression for u^3. *Hint:* Type "help(solve);".

f) Let $1, w, w^2$ be the three complex cube roots of 1. What are w and w^2? *Hint:* Solve the appropriate equation. Also, type "help(eval);", "help(evalc);", and "help(evalf);".

g) Let $y_1 = u + v$, $y_2 = wu + w^2v$, and $y_3 = w^2u + wv$. Verify that y_1, y_2, y_3 are the three roots of $g(y)$.

h) Using the "solve" command, solve the equation $g(y) = 0$ directly. How do the solutions compare with those derived in parts (e) and (g)?

i) Using the "plot3d" command, graph the polynomial $g(y)$ in various interesting ways of your choice. For example, for several interesting values of q and r, you might graph $||g(y)||$, $\mathrm{Re}(g(y))$, and $\mathrm{Im}(g(y))$, with $y = \alpha + \beta i$ ranging over the complex plane. You might also try graphing $g(y)$ over the real (y, q)-plane for various values of r. Similarly, you might graph $g(y)$ over the real (y, r)-plane for various values of q. Hand in a postscript printing of your graphs. What phenomena do you observe in your graphs? Hand in a transcript of your *Maple* session.

[1] Bruce W. Char, K. Geddes, G. Gonnet, B. Leong, M. Monagan, and S. Watt, *First Leaves: A Tutorial Introduction to Maple V*, Springer-Verlag (1992).

Problem 2: Generating Functions

In this problem you will show how to convert any summation into a recurrence, and you will solve a particular recurrence using the method of generating functions. Throughout, you are encouraged to use *Maple* as a tool to faciliate your work.

Let n and k be any positive integers, and consider the following summation:

$$S(n) = \sum_{i=1}^{n} i^k. \qquad (2)$$

a) Express $S(n)$ as a recurrence relation in n.

b) For $k = 2$, solve your recurrence from Part (a) using the method of generating functions. Express your answer as a closed-form expression for $S(n)$ in terms of n.

c) Check you answer to Part (b) by substitution.

d) What can you say about the closed-form expression for Equation 2 for $k > 2$?