Assignment 6

Problem 1: Dihedral Group (10 points)
Let $\mathcal{D}_4 = (D_4, \circ, e)$ denote the dihedral group of symmetries of a square.

a) List all elements of $\mathcal{D}_4$ in cycle notation. Give an intuitive geometric interpretation of each element.

b) Write down an addition table for the group operation $\circ$.

c) List all subgroups of $\mathcal{D}_4$. How many are there? *Warning: Be sure to identify all subgroups!*

d) Is $\mathcal{D}_4$ abelian?

e) Using concrete examples involving $\mathcal{D}_4$, explain the concepts of *cosets*, *orbits*, and *stabilizers*.

Problem 2 (Inclusion/Exclusion)
Let n and k be any positive integers. Consider the following equation involving four unknown variables x_1, x_2, x_3, and x_4:

$$x_1 + x_2 + x_3 + x_4 = n \tag{1}$$

For any positive integer k, let $\psi(n, k)$ denote the number of solutions to Equation 1 such that $x_j \leq k$ for all $1 \leq j \leq 4$. Note that each solution is a 4-tuple of integers (x_1, x_2, x_3, x_4).

a) Give a closed-form expression for $\psi(n, k)$ in terms of n and k. Carefully justify your answer.

b) Compute $\psi(7, 20)$.

Problem 3 (Backgammon)
Solve each of the following backgammon problems. Carefully justify your answers with detailed quantitative arguments.

a)

b)

c)

d)