Assignment 4

Problem 1: Diagonalization
Let $\mathcal{F}$ denote the set of *injective* functions mapping $\mathbb{N}$ to $\mathbb{N}$. Prove that $\mathcal{F}$ is uncountable.

Problem 2: Countability
For any $k \in \mathbb{N}$, let $\mathbb{N}^k$ denote the k-fold Cartesian product of $\mathbb{N}$. Is $\mathbb{N}^k$ countable? Carefully prove your answer.

Problem 3: Field Homomorphisms
Prove: Every field homomorphism is injective.

Definition (Field Homomorphism).
Let $\mathcal{F} = (F, +, *, O_F, 1_F)$ and $\mathcal{K} = (K, \oplus, \circledast, O_K, 1_K)$ be any fields, and let $g : F \rightarrow K$ be any function. We say that g is a *field homomorphism from $\mathcal{F}$ to $\mathcal{K}$* if and only if each of the following four conditions holds:

1. $g(0_F) = 0_K$, (g maps zero to zero)
2. $g(1_F) = 1_K$, (g maps one to one)
3. for all $x, y \in F$, $g(x + y) = g(x) \oplus g(y)$, and ($g$ is additive)
4. for all $x, y \in F$, $g(x * y) = g(x) \circledast g(y)$. ($g$ is multiplicative)

For the definition of a field, see handout. *Warning:* Note that $x + (-y) = 0_F$ does not immediately follow from $g(x + (-y)) = 0_K$ and $g(0_F) = 0_K$. (Why?)