Assignment 2

Note: For Problems 1–2, use the statements of induction and well-ordering given in this handout.

Problem 1: Equivalence proof

a) Prove that the weak form of mathematical induction is equivalent to the well-ordering principle.

b) Give an example of a nonempty set that has no least element.

Problem 2: Strong and weak forms of induction
Prove that the strong form of mathematical induction logically implies the weak form of mathematical induction. Does the weak form imply the strong form?

We say that statement P *logically implies* statement Q iff Q follows from P using only rules of logic (and not using model-specific information or assumptions concerning P and Q).

Problem 3: Induction Proof
Let α and β be any nonnegative real numbers, and let $T : \mathbb{N} \rightarrow \mathbb{N}$ be any function that satisfies the recurrence

$$T(n) = \begin{cases} 1 & \text{if } n < 2 \\ T(\lfloor \alpha n \rfloor) + T(\lceil \beta n \rceil) + n & \text{if } n \geq 2. \end{cases} \tag{1}$$

Prove by induction: if $\alpha + \beta < 1$, then there exists some real number c and some natural number N_0 such that, for all $n \geq N_0$, $T(n) \leq cn$.

Principle of Mathematical Induction (weak form)
Let S be any subset of natural numbers.[1] *If*

1. $0 \in S$, *and*

2. *for every* $n \in \mathbb{N}$, $n \in S$ *implies* $n + 1 \in S$,

then $S = \mathbb{N}$.

Note: Whenever doing a formal induction proof, for clarity and specificity, you should explicitly identify the set S. For example, this set might be the set of array sizes for which the algorithm works correctly.

Principle of Mathematical Induction (strong form)
Let S be any subset of natural numbers. If

1. $0 \in S$, *and*

2. *for every* $n \in \mathbb{N}$, $\{0, 1, 2, \ldots, n\} \subseteq S$ *implies* $n + 1 \in S$,

then $S = \mathbb{N}$.

The Well-Ordering Principle
Every nonempty subset of natural numbers has a least element.

[1] Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ denote the set of natural numbers.