Let's consider the problem from the book in which we don't know whether it's raining, but we know whether we've seen an umbrella. We also know a sensor model – how likely we are to see an umbrella if it's raining – and we know a transition model – how likely it is to rain if it rained the day before. (Here, our timesteps = days.)

Transition Model / Sensor Model

R_{t-1}	$P(R_t \mid R_{t-1})$
T	0.7
F	0.3

R_t	$P(U_t \mid R_t)$
T	0.9
F	0.2

So if it rains on day *n*, there's a 70% chance it will rain on day *n*+1. If it's raining, the probability of seeing an umbrella is 90%. If it's NOT raining, the probability of seeing an umbrella is 20%.

So is it raining on day 2, if we saw umbrellas on days 1 and 2?

The basic idea is that, for each day, you can work out the probability that it is raining, given (1) the evidence (whether you see an umbrella), and (2) whether it rained the day before. But you don't know if it rained the day before! So, you can work out whether it rained the day before given *that* day's umbrella sighting and the rain from the day before *that*... and so on recursively back to the beginning of time, when you have some distribution over whether it "started out" raining (the prior). So you are always working out the probability of R_t for a single day, t, working forward to the final day. The "recursive factors" are the information from the previous days. So let's do the math forward instead of trying to do it recursively.

$P(R_1)$

First, we predict whether it rained the first day. We give it 50/50 odds that it started out raining, so the prior is <0.5, 0.5> (that is, P(R=t) = 0.5, P(R=f) = 0.5.) Before we consider any evidence, we already have a belief, P(Rain) = <Rain = true, Rain=false>. We sum over the possible values for R_0 (true or false):

$$\begin{aligned}\mathbf{P}(\mathbf{R}_1) &= \Sigma r_0\, \mathbf{P}(R_1 \mid r_0)\, \mathbf{P}(r_0) \\ &= \langle 0.7,0.3\rangle \times 0.5 + \langle 0.3,0.7\rangle \times 0.5 \\ &= \langle 0.35,0.15\rangle + \langle 0.15,0.35\rangle \\ &= \langle 0.5,0.5\rangle\end{aligned}$$

This is what we intuitively expected. We haven't made any umbrella observations yet, so our **prediction** of it raining hasn't changed from our initial guess. So this gives us the probability of rain when we don't have any evidence.

$P(R_1|U_1)$

But once we observe an umbrella, U_1 = true, so we can update our belief based on what we just calculated, PLUS our model of rain causing umbrella sightings (our sensor model). We start with Bayes rule and then plug in numbers from the tables and from the prediction we just made:

$$\begin{aligned}\mathbf{P}(\mathbf{R}_1|U_1) &= \alpha\mathbf{P}(u_1|R_1)\mathbf{P}(R_1)\\ &= \alpha\langle 0.9,0.2\rangle\langle 0.5,0.5\rangle\\ &= \alpha\langle 0.45,0.1\rangle\\ &\approx \langle 0.818,0.182\rangle\end{aligned}$$

So, since umbrella is strong evidence for rain, the probability of rain is much higher once we take the observation into account: P(rain on day 1 = true) = 0.818, P(rain on day 1 = false) = 0.182. (Alpha is just a normalizing constant that makes the probabilities add up to 1. We get it by dividing *each* element by the sum of *both* elements, e.g., 0.45/(0.45+0.1) ≈ 0.818.)

$P(R_2|U_1)$

We can then carry out the same computation for Day 2. First we predict whether it will rain given (1) what we know from the previous day and (2) our belief about whether it is likely to rain given the previous day's weather (the transition model). We're summing over R_1 this time:

$$\begin{aligned}\mathbf{P}(\mathbf{R}_2|u_1) &= \Sigma r_1\ \mathbf{P}(R_2|r_1)\mathbf{P}(r_1|u_1)\\ &= \langle 0.7,0.3\rangle \times 0.818 + \langle 0.3,0.7\rangle \times 0.182\\ &\approx \langle 0.627,0.373\rangle\end{aligned}$$

So even without evidence on the second day there is a more than 50/50 probability of rain, because rain tends to follow rain, and we have an increased belief that it rained on R_1 because we saw an umbrella.

$P(R_2|U_1,U_2)$

But then we take into account the fact that we DID see an umbrella at t=2, which we combine with our pre-observation expectation about whether it's raining (what we just calculated):

$$\begin{aligned}\mathbf{P}(\mathbf{R}_2|u_1,u_2) &= \alpha\mathbf{P}(u_2|R_2)\mathbf{P}(R_2|u_1)\\ &= \alpha\langle 0.9, 0.2\rangle\langle 0.627, 0.373\rangle\\ &= \alpha\langle 0.565, 0.075\rangle\\ &\approx \langle 0.883, 0.117\rangle\end{aligned}$$

And this is the correct answer for "What is the probability of rain on day 2, given umbrella sightings on day 1 and day 2" – **$\mathbf{P(R_2=t) = 0.883}$**. The whole idea of factors is that you can start from the last calculation and work backwards, recursively evaluating for previous timesteps as they come up in the calculation, but doing it forward works for small values of t.