Propositional Logic

Chapter 7.4–7.8

Some material adopted from notes by Andreas Geyer-Schulz and Chuck Dyer

Propositional logic syntax

- Users specify
  - Set of propositional symbols (e.g., P, Q) whose values can be **True** or **False**
  - What each *means*, e.g.: P: *"It's hot"*, Q: *"It's humid"*
- A sentence (well formed formula) is defined as:
  - Any symbol is a sentence
  - If S is a sentence, then **$\neg$S** is a sentence
  - If S is a sentence, then **(S)** is a sentence
  - If S and T are sentences, then so are **(S $\vee$ T), (S $\wedge$ T), (S $\rightarrow$ T)**, and **(S $\leftrightarrow$ T)**
  - A finite number of applications of the rules

Examples of PL sentences

- **Q**

  "It's humid"

- **$Q \rightarrow P$**

  "If it's humid, then it's hot"

- **$(P \wedge Q) \rightarrow R$**

  "If it's hot and it's humid, then it's raining"

- We're free to choose better symbols, e.g.:

  Hot for "It's hot"

  Humid for "It's humid"

  Raining for "It's raining"

Some terms

- Given the truth values of all symbols in a sentence, it can be ***evaluated*** to determine its **truth value** (True or False)
- We consider a **Knowledge Base** (KB) to be a set of sentences that are all True
- A **model** for a KB is a **possible world** – an assignment of truth values to propositional symbols that makes each KB sentence true

A simple example

The KB

P

Q ∨ ¬ R

The KB has 2 sentences.

The KB has 3 variables.

Models for the KB

	P	Q	R
#1	T	T	F
#2	T	T	T
#3	T	F	T

The KB has 3 models. Each model has a value for every variable in the KB such every sentence evaluates to true.

Another simple example

The KB

$P \wedge Q$

$R \wedge \neg P$

The KB has 2 sentences.

The KB has 3 variables.

Models for the KB

P	Q	R

The KB has no models. There is no assignment of True or False to every variable that makes every sentence in the KB true

More terms

- A **valid sentence** or **tautology:** one that's **True** under all interpretations, no matter what the world is actually like or what the semantics is. Example: "It's raining or it's not raining" $(P \vee \neg P)$

- An **inconsistent sentence** or **contradiction**: a sentence that's **False** under all interpretations. The world is never like what it describes, as in "It's raining and it's not raining." $(P \wedge \neg P)$

Truth tables

Used to define meaning of logical connectives

Truth tables for the five logical connectives

P	Q	$\neg P$	$P \wedge Q$	$P \vee Q$	$P \Rightarrow Q$	$P \Leftrightarrow Q$
False	*False*	*True*	*False*	*False*	*True*	*True*
False	*True*	*True*	*False*	*True*	*True*	*False*
True	*False*	*False*	*False*	*True*	*False*	*False*
True	*True*	*False*	*True*	*True*	*True*	*True*

Given a value for P and for Q, the truth table defines the value of P∨Q

Truth tables

Used to define meaning of logical connectives **and to determine when a complex sentence is true given values of its symbols**

The value of complex sentences can be determined from the values of their elements

Truth tables for the five logical connectives

P	Q	$\neg P$	$P \wedge Q$	$P \vee Q$	$P \Rightarrow Q$	$P \Leftrightarrow Q$
False	False	True	False	False	True	True
False	True	True	False	True	True	False
True	False	False	False	True	False	False
True	True	False	True	True	True	True

Example of a truth table used for a complex sentence

P	H	$P \vee H$	$(P \vee H) \wedge \neg H$	$((P \vee H) \wedge \neg H) \Rightarrow P$
False	False	False	False	True
False	True	True	False	True
True	False	True	True	True
True	True	True	False	True

The implies connective: $P \rightarrow Q$

$\rightarrow$ is a *logical connective*

- $P \rightarrow Q$ is a **logical sentence** and has a truth value, i.e., is either **True** or **False**
- If the sentence is in a KB, it can be used by a rule (*Modes Ponens*) to infer that Q is True if P is True in the KB
- Given a KB where P=True and Q=True, we can derive/infer/prove that $P \rightarrow Q$ is True
- Note: $P \rightarrow Q$ is equivalent to $\sim P \vee Q$

$P \to Q$

When is $P \to Q$ true? Check all that apply

- ❑ P=Q=true
- ❑ P=Q=false
- ❑ P=true, Q=false
- ❑ P=false, Q=true

$P \rightarrow Q$

When is *P→Q* true? Check all that apply

- [x] P=Q=true
- [x] P=Q=false
- [ ] P=true, Q=false
- [x] P=false, Q=true

- We can get this from the truth table for $\rightarrow$
- Note: in FOL it's much harder to prove that a conditional true, e.g., prime(x) $\rightarrow$ odd(x)

$P \rightarrow Q \equiv \sim P \vee Q$

- $P \rightarrow Q$ is equivalent to $\sim P \vee Q$
- We can show this by looking at a truth table

P	Q	$P \rightarrow Q$	$\sim P \vee Q$
F	F	T	T
F	T	T	T
T	F	F	F
T	T	T	T

These two columns are the same

Models for a KB

- KB: [PvQ, P→R, Q→R]
- What are the sentences?

  s1: PVQ
  s2: P→R
  s3: Q→R
- What are the propositional variables?

  P, Q, R
- What are the candidate models?
  1) Consider all **eight** possible assignments of T|F to P, Q, R
  2) Check if each sentence is consistent with the model

P	Q	R	s1	s2	s3
F	F	F	x	✓	✓
F	F	T	x	✓	✓
F	T	F	✓	✓	x
F	T	T	✓	✓	✓
T	F	F	✓	x	✓
T	F	T	✓	✓	✓
T	T	F	✓	x	x
T	T	T	✓	✓	✓

Here **x** means the model makes the sentence False and ✓ means it doesn't make it False

Models for a KB

- KB: [PvQ, P→R, Q→R]
- What are the sentences?

  s1: PVQ
  s2: P→R
  s3: Q→R
- What are the propositional variables?

  P, Q, R
- What are the candidate models?
  1) Consider all possible assignments of T|F to P, Q, R
  2) Check truth tables for consistency, eliminating any row that does not make every KB sentence true

P	Q	R	s1	s2	s3
~~F~~	~~F~~	~~F~~	~~x~~	~~✓~~	~~✓~~
~~F~~	~~F~~	~~T~~	~~x~~	~~✓~~	~~✓~~
~~F~~	~~T~~	~~F~~	~~✓~~	~~✓~~	~~x~~
F	T	T	✓	✓	✓
~~T~~	~~F~~	~~F~~	~~✓~~	~~x~~	~~✓~~
T	F	T	✓	✓	✓
~~T~~	~~T~~	~~F~~	~~✓~~	~~x~~	~~x~~
T	T	T	✓	✓	✓

- Only 3 models are consistent with KB
- R true in all of them
- Therefore R is true and can be added to the KB

Fin