Propositional and First-Order Logic

Chapter 7.4–7.8, 8.1–8.3, 8.5

Some material adopted from notes by Andreas Geyer-Schulz and Chuck Dyer

Logic roadmap overview

- Propositional logic
  - Problems with propositional logic
- First-order logic
  - Properties, relations, functions, quantifiers, …
  - Terms, sentences, wffs, axioms, theories, proofs, …
  - Extensions to first-order logic
- Logical agents
  - Reflex agents
  - Representing change: situation calculus, frame problem
  - Preferences on actions
  - Goal-based agents

Disclaimer

"Logic, like whiskey, loses its beneficial effect when taken in too large quantities."

- *Lord Dunsany*

Propositional Logic: Review

Big Ideas

- Logic is a great knowledge representation language for many AI problems
- **Propositional logic** is the simple foundation and fine for many AI problems
- **First order logic** (FOL) is much more expressive as a knowledge representation (KR) language and more commonly used in AI
- **Variations** on FOL are common: horn logic, higher order logic, three-valued logic, probabilistic logic, fuzzy logic, etc.

Propositional logic syntax

- **Logical constants**: true, false
- **Propositional symbols**: P, Q, ... (aka **atomic sentences**)
- **Parentheses**: (…)
- **Sentences** are build with **connectives**:

$\wedge$	and	[conjunction]
$\vee$	or	[disjunction]
$\Rightarrow$	implies	[implication/conditional/if]
$\Leftrightarrow$	is equivalent	[biconditional/iff]
$\neg$	not	[negation]

- **Literal**: atomic sentence or their negation: P, $\neg$P

Propositional logic syntax

- Simplest logic language in which a user specifies
  - Set of propositional symbols (e.g., P, Q)
  - What each *means*, e.g.: P: "*It's hot*", Q: "*It's humid*"
- A sentence (well formed formula) is defined as:
  - Any symbol is a sentence
  - If S is a sentence, then $\neg$**S** is a sentence
  - If S is a sentence, then **(S)** is a sentence
  - If S and T are sentences, then so are **(S $\vee$ T), (S $\wedge$ T), (S $\rightarrow$ T),** and **(S $\leftrightarrow$ T)**
  - A sentence results from a finite number of applications of the rules

Examples of PL sentences

- $(P \wedge Q) \rightarrow R$

  "If it is hot and humid, then it is raining"
- $Q \rightarrow P$

  "If it is humid, then it is hot"
- Q

  "It is humid."
- We're free to choose better symbols, e.g.:

  Hot = "It is hot"

  Humid = "It is humid"

  Raining = "It is raining"

Some terms

- The meaning or **semantics** of a sentence determines its **interpretation**
- Given the truth values of all symbols in a sentence, it can be ***evaluated*** to determine its **truth value** (True or False)
- A **model** for a KB is a *possible world* – an assignment of truth values to propositional symbols that makes each KB sentence true

Models for a KB

- KB: [$P \wedge Q \rightarrow R$, $Q \rightarrow P$]
- What are the sentences?
  - $P \wedge Q \rightarrow R$
  - $Q \rightarrow P$
- What are the propositional variables?
  - P, Q, R
- What are the possible models?
  - Consider all possible assignments of T|F to P, Q, R
  - Check truth tables for consistency

P: it's hot
Q: it's humid
R: it's raining

P Q R	
F F F	: OK
F F T	: OK
F T F	: NO
F T T	: NO
T F F	: OK
T F T	: OK
T T F	: NO
T T T	: OK

Models for a KB

- Add Q to the KB
- KB: [$P \wedge Q \rightarrow R$, $Q \rightarrow P$, Q]
- What are the sentences?
  - $P \wedge Q \rightarrow R$
  - $Q \rightarrow P$
  - Q
- What are the propositional variables?
  - P, Q, R
- What are the possible models?
  - Consider all possible assignments of T|F to P, Q, R
  - Check truth tables for consistency

P: it's hot
Q: it's humid
R: it's raining

P	*Q*	*R*	
F	F	F	: OK
F	F	T	: OK
F	T	F	: NO
F	T	T	: NO
T	F	F	: NO
T	F	T	: NO
T	T	F	: NO
T	**T**	**T**	**: OK**

Since P & R are true in every KB model, the KB entails that P & R are True

More terms

- A **valid sentence** or **tautology** is a sentence that's **True** under all interpretations, no matter what the world is actually like or what the semantics is. Example: "It's raining or it's not raining"
- An **inconsistent sentence** or **contradiction** is a sentence that's **False** under all interpretations. The world is never like what it describes, as in "It's raining and it's not raining."
- **P entails Q**, written P |= Q, means that whenever P is True, so is Q
  - In all models in which P is true, Q is also true

Truth tables

- Truth tables are used to define meaning of logical connectives
- And to determine when a complex sentence is true given the values of the symbols in it

Truth tables for the five logical connectives

P	Q	$\neg P$	$P \wedge Q$	$P \vee Q$	$P \Rightarrow Q$	$P \Leftrightarrow Q$
False	*False*	*True*	*False*	*False*	*True*	*True*
False	*True*	*True*	*False*	*True*	*True*	*False*
True	*False*	*False*	*False*	*True*	*False*	*False*
True	*True*	*False*	*True*	*True*	*True*	*True*

Example of a truth table used for a complex sentence

P	H	$P \vee H$	$(P \vee H) \wedge \neg H$	$((P \vee H) \wedge \neg H) \Rightarrow P$
False	*False*	*False*	*False*	*True*
False	*True*	*True*	*False*	*True*
True	*False*	*True*	*True*	*True*
True	*True*	*True*	*False*	*True*

The implies connective: $P \rightarrow Q$

- $\rightarrow$ is a *logical connective*
- So $P \rightarrow Q$ is a **logical sentence** and has a truth value, i.e., is either true or false
- If we add this sentence to a KB, it can be used by an inference rule, *Modes Ponens*, to derive/infer/prove Q if P is also in the KB
- Given a KB where P=True and Q=True, we can derive/infer/prove that $P \rightarrow Q$ is True
- Note: $P \rightarrow Q$ is equivalent to $\sim P \vee Q$

$P \rightarrow Q$

- When is $P \rightarrow Q$ true? Check all that apply
  - ❑ P=Q=true
  - ❑ P=Q=false
  - ❑ P=true, Q=false
  - ❑ P=false, Q=true

$P \to Q$

- When is $P \to Q$ true? Check all that apply
  - ☑ P=Q=true
  - ☑ P=Q=false
  - ☐ P=true, Q=false
  - ☑ P=false, Q=true
- We can get this from the truth table for $\to$
- Note: in FOL it's much harder to prove that a conditional true, e.g., prime(x) $\to$ odd(x)

Inference rules

- **Logical inference** creates new sentences that logically follow from a set of sentences (KB)
- An inference rule is **sound** if every sentence X it produces when operating on a KB logically follows from the KB
  - i.e., inference rule creates no contradictions
- An inference rule is **complete** if it can produce every expression that logically follows from (is entailed by) the KB
  - Note analogy to complete search algorithms

Sound rules of inference

Here are examples of sound rules of inference

Each can be shown to be sound using a truth table

RULE	PREMISE	CONCLUSION
Modus Ponens	$A, A \rightarrow B$	B
And Introduction	A, B	$A \wedge B$
And Elimination	$A \wedge B$	A
Double Negation	$\neg\neg A$	A
Unit Resolution	$A \vee B, \neg B$	A
Resolution	$\mathbf{A \vee B, \neg B \vee C}$	$\mathbf{A \vee C}$

Resolution

- **Resolution** is a valid inference rule producing a new clause implied by two clauses containing *complementary literals*

  Literal: atomic symbol or its negation, i.e., P, ~P

- Amazingly, this is the only interference rule needed to build a sound & complete theorem prover
  - Based on proof by contradiction, usually called resolution refutation
- The resolution rule was discovered by Alan Robinson (CS, U. of Syracuse) in the mid 1960s

Resolution

- A KB is a set of sentences all of which are true, i.e., a conjunction of sentences
- To use resolution, put KB into *conjunctive normal form* (CNF)
  - Each sentence is a disjunction of one or more literals (positive or negative atoms)
- Every KB can be put into CNF, it's just a matter of rewriting its sentences using standard tautologies, e.g.:
  - $P \rightarrow Q \equiv \sim P \vee Q$

Resolution Example

Tautologies

$(A \rightarrow B) \leftrightarrow (\sim A \vee B)$

$(A \vee (B \wedge C)) \leftrightarrow$

$(A \vee B) \wedge (A \vee C)$

- KB: $[P \rightarrow Q\ ,\ Q \rightarrow R \wedge S]$
- KB: $[P \rightarrow Q\ ,\ Q \rightarrow R,\ Q \rightarrow S\ ]$
- KB in CNF: $[\sim P \vee Q\ ,\ \sim Q \vee R\ ,\ \sim Q \vee S]$
- Resolve KB[0] and KB[1] producing:

  $\sim P \vee R$ *(i.e., $P \rightarrow R$)*
- Resolve KB[0] and KB[2] producing:

  $\sim P \vee S$ *(i.e., $P \rightarrow S$)*
- New KB: $[\sim P \vee Q\ ,\ \sim Q \vee R,\ \sim Q \vee S,\ \sim P \vee R,\ \sim P \vee S]$

Soundness of resolution inference rule

α	β	γ	$\alpha \vee \beta$	$\neg\beta \vee \gamma$	$\alpha \vee \gamma$
False	*False*	*False*	*False*	*True*	*False*
False	*False*	*True*	*False*	*True*	*True*
False	*True*	*False*	*True*	*False*	*False*
<u>*False*</u>	<u>*True*</u>	<u>*True*</u>	<u>*True*</u>	<u>*True*</u>	<u>*True*</u>
<u>*True*</u>	<u>*False*</u>	<u>*False*</u>	<u>*True*</u>	<u>*True*</u>	<u>*True*</u>
<u>*True*</u>	<u>*False*</u>	<u>*True*</u>	<u>*True*</u>	<u>*True*</u>	<u>*True*</u>
True	*True*	*False*	*True*	*False*	*True*
<u>*True*</u>	<u>*True*</u>	<u>*True*</u>	<u>*True*</u>	<u>*True*</u>	<u>*True*</u>

From rightmost three columns of truth table, we see that

$$(\boldsymbol{\alpha} \vee \boldsymbol{\beta}) \wedge ({\sim}\boldsymbol{\beta} \vee \boldsymbol{\gamma}) \rightarrow (\boldsymbol{\alpha} \vee \boldsymbol{\gamma})$$

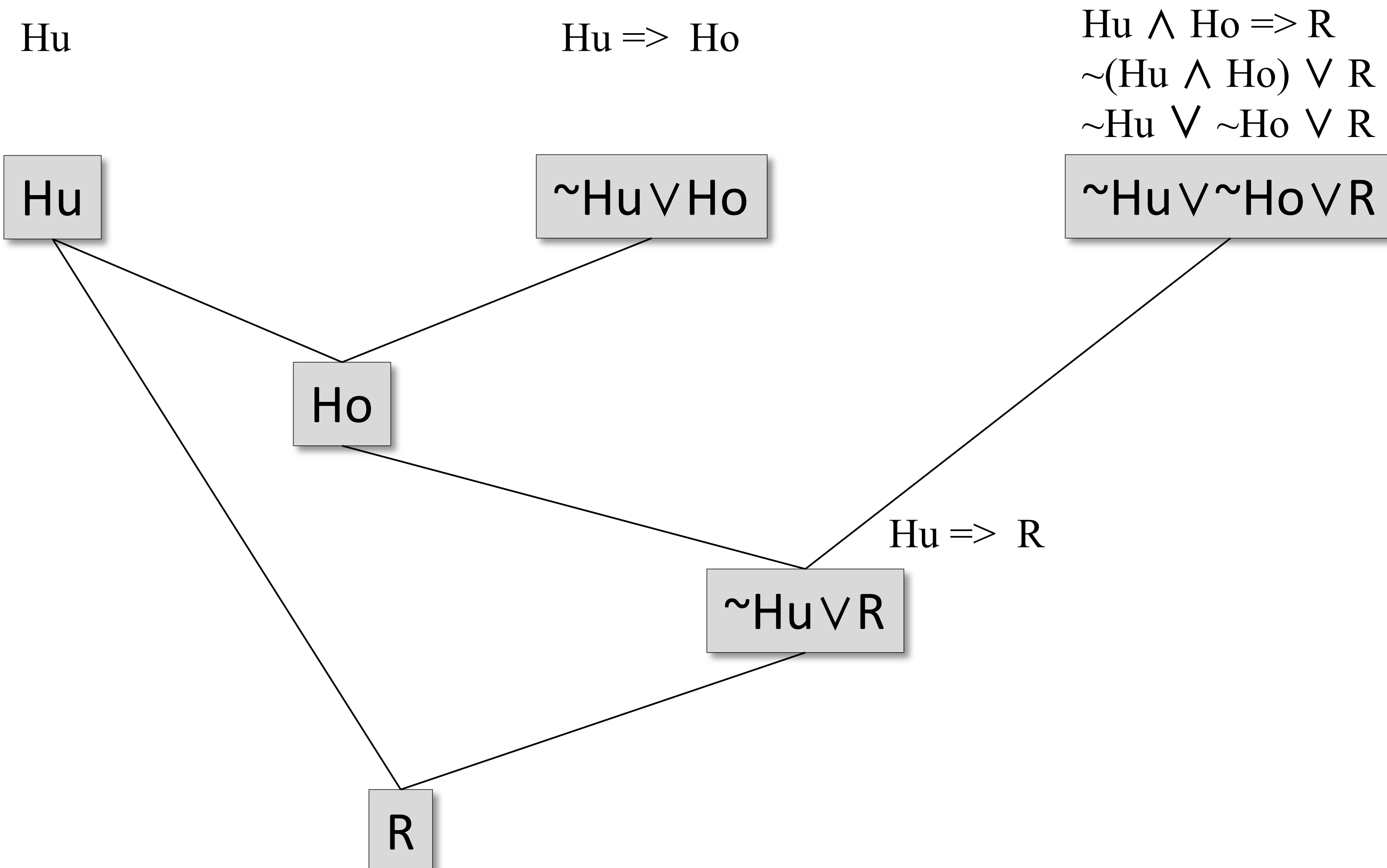
is valid (i.e., always true regardless of truth values for α, β and γ

Proving it's raining (1)

- A **proof** is a sequence of sentences, where each is a premise (i.e., a given) or is derived from earlier sentences in the proof by an inference rule
- Last sentence is the **theorem** (also called goal or query) that we want to prove
- The *weather problem* using traditional reasoning

1 Hu	premise	"It's humid"
2 $Hu \rightarrow Ho$	premise	"If it's humid, it's hot"
3 Ho	modus ponens(1,2)	"It's hot"
4 $(Ho \wedge Hu) \rightarrow R$	premise	"If it's hot & humid, it's raining"
5 $Ho \wedge Hu$	and introduction(1,3)	"It's hot and humid"
6 R	modus ponens(4,5)	"It's raining"

Proving it's raining (2)



A simple proof procedure

This procedure generates new sentences from a KB

1. Convert all sentences in the KB to CNF
2. Find all pairs of sentences in KB with complementary literals that have not yet been resolved
3. If there are no pairs stop else resolve each pair, adding the result to the KB and go to 2

- Is it sound?
- Is it complete?
- Will it always terminate?

Resolution refutation

1. Add negation of goal to the KB
2. Convert all sentences in KB to CNF
3. Find all pairs of sentences in KB with complementary literals that have not yet been resolved
4. If there are no pairs stop else resolve each pair, adding the result to the KB and go to 2

- If we derived an empty clause (i.e., a contradiction) then the conclusion follows from the KB
- If we did not, the conclusion cannot be proved from the KB

Horn* sentences

- A **Horn sentence** or **Horn clause** has the form:

  $P1 \wedge P2 \wedge P3 \ldots \wedge Pn \rightarrow Qm$ *where n>=0, m in{0,1}*

- Note: a conjunction of 0 or more symbols to left of $\rightarrow$ and 0-1 symbols to right
- Special cases:
  - n=0, m=1: **P** *(assert P is true)*
  - n>0, m=0: **P∧Q** $\rightarrow$ *(constraint: both P and Q can't be true)*
  - n=0, m=0: *(well, there is nothing there!)*
- Put in CNF: each sentence is a disjunction of literals with at most one non-negative literal

  $\neg P1 \vee \neg P2 \vee \neg P3 \ldots \vee \neg Pn \vee Q$

$\boxed{(P \rightarrow Q) = (\neg P \vee Q)}$

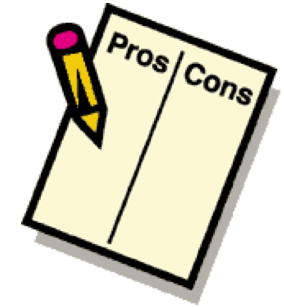
* After Alfred Horn

Significance of Horn logic

- We can also have horn sentences in FOL
- Reasoning with horn clauses is much simpler
  - Satisfiability of propositional KB (i.e., finding values for a symbols that will make it true) is NP complete
  - Restricting KB to horn sentences, satisfiability is in P
- For this reason, FOL Horn sentences are the basis for many rule-based languages, including Prolog and Datalog
- Horn logic can't handle, in a general way, **negation** and **disjunctions**

Problems with Propositional Logic

Propositional logic: pro and con



- **Advantages**
  - Simple KR language good for many problems
  - Lays foundation for higher logics (e.g., FOL)
  - Reasoning is decidable, though NP complete; efficient techniques exist for many problems
- **Disadvantages**
  - Not expressive enough for most problems
  - Even when it is, it can very "un-concise"

PL is a weak KR language

- Hard to identify *individuals* (e.g., Mary, 3)
- Can’t directly represent properties of individuals or relations between them (e.g., “Bill is tall”)
- Generalizations, patterns, regularities hard to represent (e.g., “all triangles have 3 sides”)
- First-Order Logic (FOL) represents this information via **relations**, **variables** & **quantifier**s, e.g.,
  - *Every elephant is gray:* $\forall$ x (elephant(x) → gray(x))
  - *There is a black swan:* $\exists$ x (swan(X) ^ black(X))

PL Example

- Consider the problem of representing the following information:
  - Every person is mortal.
  - Confucius is a person.
  - Confucius is mortal.
- How can these sentences be represented so that we can infer the third sentence from the first two?

PL Example

- In PL we must create propositional symbols to stand for all or part of each sentence, e.g.:

  P = "person"; Q = "mortal"; R = "Confucius"
- The above 3 sentences are represented as:

  $P \rightarrow Q$; $R \rightarrow P$; $R \rightarrow Q$
- The 3rd sentence is entailed by the first two, but we need an explicit symbol, R, to represent an individual, Confucius, who is a member of the classes *person* and *mortal*
- Representing other individuals requires introducing separate symbols for each, with some way to represent the fact that all individuals who are "people" are also "mortal"

Hunt the Wumpus domain

- Some atomic propositions:

  S12 = There is a stench in cell (1,2)

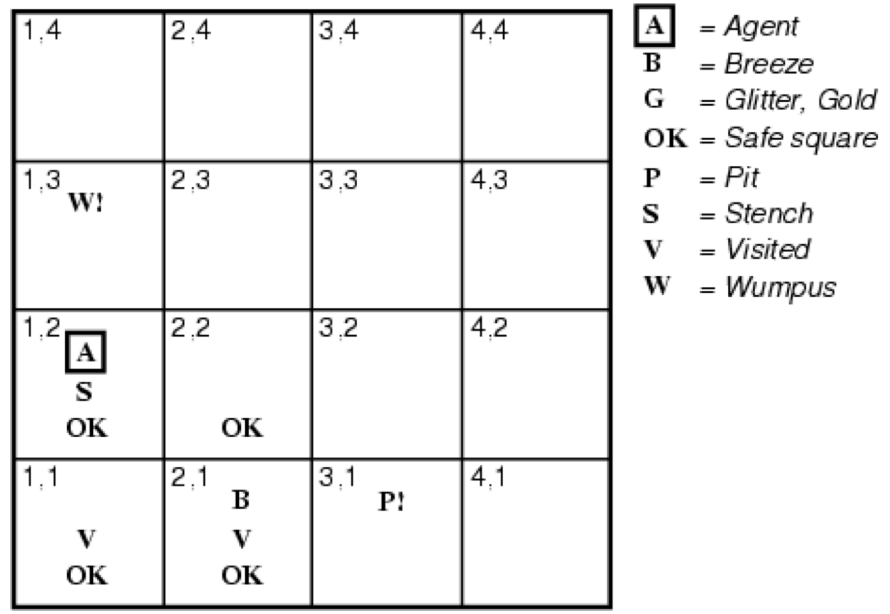
  B34 = There is a breeze in cell (3,4)

  W22 = Wumpus is in cell (2,2)

  V11 = We've visited cell (1,1)

  OK11 = Cell (1,1) is safe

  …



- Some rules:

  $\neg S22 \rightarrow \neg W12 \wedge \neg W23 \wedge \neg W32 \wedge \neg W21$

  $S22 \rightarrow W12 \vee W23 \vee W32 \vee W21$

  $B22 \rightarrow P12 \vee P23 \vee P32 \vee P21$

  $W22 \rightarrow S12 \wedge S23 \wedge S23 \wedge W21$

  $W22 \rightarrow \neg W11 \wedge \neg W21 \wedge \ldots \neg W44$

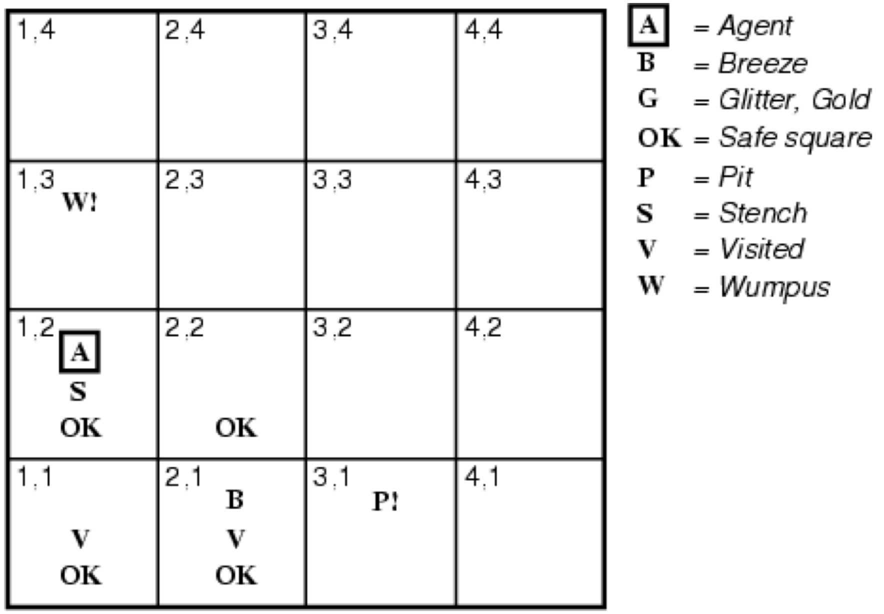
  $A22 \rightarrow V22$

  $A22 \rightarrow \neg W11 \wedge \neg W21 \wedge \ldots \neg W44$

  $V22 \rightarrow OK22$

Hunt the Wumpus domain

- Eight variables for each cell, i.e.: A11, B11, G11, OK11, P11, S11, V11, W11
- Lack of variables requires giving similar rules for each cell!
- Ten rules (I think) for each



$A11 \rightarrow \ldots$

$V11 \rightarrow \ldots$

$P11 \rightarrow \ldots$

$\neg P11 \rightarrow \ldots$

$W11 \rightarrow \ldots$

$\neg W11 \rightarrow \ldots$

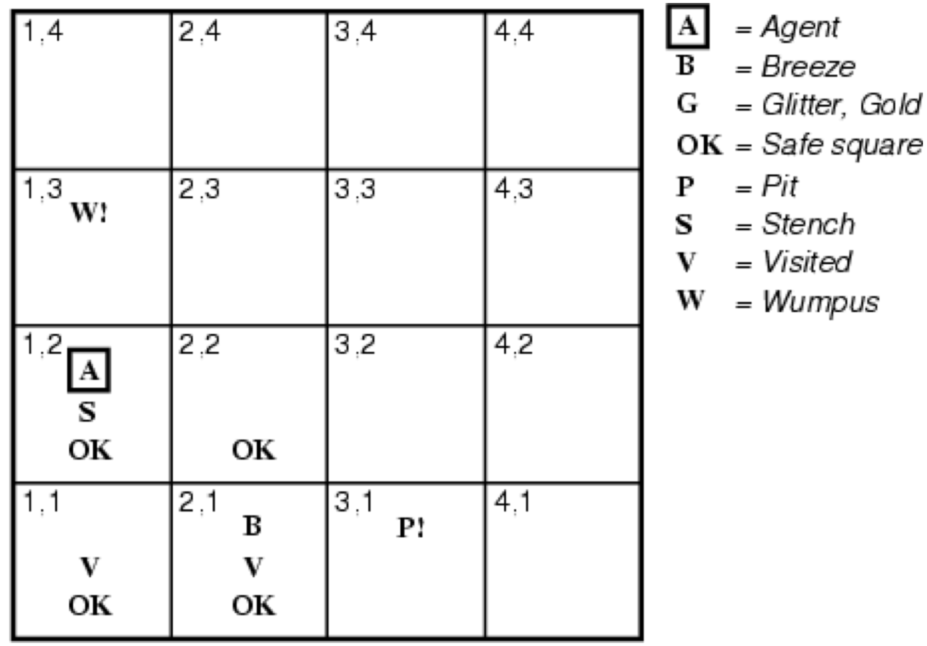
$S11 \rightarrow \ldots$

$\neg S11 \rightarrow \ldots$

$B11 \rightarrow \ldots$

$\neg B11 \rightarrow \ldots$

After third move



- We can prove that the Wumpus is in (1,3) using these four rules
- See R&N section 7.5

(R1) $\neg S11 \rightarrow \neg W11 \wedge \neg W12 \wedge \neg W21$

(R2) $\neg S21 \rightarrow \neg W11 \wedge \neg W21 \wedge \neg W22 \wedge \neg W31$

(R3) $\neg S12 \rightarrow \neg W11 \wedge \neg W12 \wedge \neg W22 \wedge \neg W13$

(R4) $S12 \rightarrow W13 \vee W12 \vee W22 \vee W11$

Proving W13

(R1) $\neg S11 \rightarrow \neg W11 \wedge \neg W12 \wedge \neg W21$

(R2) $\neg S21 \rightarrow \neg W11 \wedge \neg W21 \wedge \neg W22 \wedge \neg W31$

(R3) $\neg S12 \rightarrow \neg W11 \wedge \neg W12 \wedge \neg W22 \wedge \neg W13$

(R4) $S12 \rightarrow W13 \vee W12 \vee W22 \vee W11$

Apply MP with $\neg S11$ and R1:

$\neg W11 \wedge \neg W12 \wedge \neg W21$

Apply And-Elimination to this, yielding 3 sentences:

$\neg W11, \neg W12, \neg W21$

Apply MP to ~S21 and R2, then apply And-elimination:

$\neg W22, \neg W21, \neg W31$

Apply MP to S12 and R4 to obtain:

$W13 \vee W12 \vee W22 \vee W11$

Apply Unit Resolution on $(W13 \vee W12 \vee W22 \vee W11)$ and $\neg W11$:

$W13 \vee W12 \vee W22$

Apply Unit Resolution with $(W13 \vee W12 \vee W22)$ and $\neg W22$:

$W13 \vee W12$

Apply Unit Resolution with $(W13 \vee W12)$ and $\neg W12$:

W13

QED

Propositional Wumpus hunter problems

- Lack of variables prevents stating more general rules, like these:
  - $\forall$ x, y V(x,y) → OK(x,y)
  - $\forall$ x, y S(x,y) → W(x-1,y) $\vee$ W(x+1,y) …
- Change of the KB over time is difficult to represent
  - In classical logic, a fact is true or false for all time
  - A standard technique is to index dynamic facts with the time when they're true
    - A(1, 1, t0)
  - Thus we have a separate KB for every time point

Propositional logic summary

- **Inference**: process of deriving new sentences from old
  - **Sound** inference derives true conclusions given true premises
  - **Complete** inference derives all true conclusions from a set of premises
- **Valid sentence:** true in all worlds under all interpretations
- If an implication sentence can be shown to be valid, then, given its premise, its consequent can be derived
- Different logics make different **commitments** about what the world is made of and the kind of beliefs we can have
- **Propositional logic** commits only to existence of facts that may or may not be the case in the world being represented
  - Simple syntax and semantics suffices to illustrate the process of inference
  - Propositional logic can become impractical, even for very small worlds