Bayesian Reasoning

Chapter 13

Thomas Bayes, 1701-1761

Today's topics

- Review probability theory
- Bayesian inference
  - From the joint distribution
  - Using independence/factoring
  - From sources of evidence

Sources of Uncertainty

- Uncertain **inputs --** missing and/or noisy data
- Uncertain **knowledge**
  - Multiple causes lead to multiple effects
  - Incomplete enumeration of conditions or effects
  - Incomplete knowledge of causality in the domain
  - Probabilistic/stochastic effects
- Uncertain **outputs**
  - Abduction and induction are inherently uncertain
  - Default reasoning, even deductive, is uncertain
  - Incomplete deductive inference may be uncertain

▸ Probabilistic reasoning only gives probabilistic results (summarizes uncertainty from various sources)

Decision making with uncertainty

Rational behavior:

- For each possible action, identify the possible outcomes
- Compute the **probability** of each outcome
- Compute the **utility** of each outcome
- Compute the probability-weighted **(expected) utility** over possible outcomes for each action
- Select action with the highest expected utility (principle of **Maximum Expected Utility**)

Why probabilities anyway?

Kolmogorov showed that three simple axioms lead to the rules of probability theory

1. All probabilities are between 0 and 1:

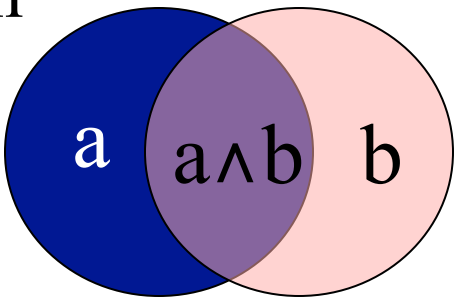
   $0 \leq P(a) \leq 1$

2. Valid propositions (tautologies) have probability 1, and unsatisfiable propositions have probability 0:

   $P(true) = 1 ; P(false) = 0$

3. The probability of a disjunction is given by:

   $P(a \vee b) = P(a) + P(b) - P(a \wedge b)$



Probability theory 101

- **Random variables**
  - Domain
- Alarm, Burglary, Earthquake
  - Boolean (like these), discrete, continuous
- **Atomic event**: complete specification of state
- Alarm=T∧Burglary=T∧Earthquake=F alarm ∧ burglary ∧ ¬earthquake
- **Prior probability**: degree of belief without any other evidence
- P(Burglary) = 0.1
  P(Alarm) = 0.1
  P(earthquake) = 0.000003
- **Joint probability**: matrix of combined probabilities of a set of variables
- P(Alarm, Burglary) =

	alarm	**¬alarm**
burglary	.09	.01
¬burglary	.1	.8

Probability theory 101

	alarm	¬alarm
burglary	.09	.01
¬burglary	.1	.8

- **Conditional probability**: prob. of effect given causes
- **Computing conditional probs**:
  - $P(a \mid b) = P(a \wedge b) / P(b)$
  - P(b): **normalizing** constant
- **Product rule**:
  - $P(a \wedge b) = P(a \mid b) * P(b)$
- **Marginalizing**:
  - $P(B) = \Sigma_a P(B, a)$
  - $P(B) = \Sigma_a P(B \mid a) P(a)$ (**conditioning**)

- $P(\text{burglary} \mid \text{alarm}) = .47$
  $P(\text{alarm} \mid \text{burglary}) = .9$
- $P(\text{burglary} \mid \text{alarm}) =$
  $P(\text{burglary} \wedge \text{alarm}) / P(\text{alarm})$
  $= .09/.19 = .47$
- $P(\text{burglary} \wedge \text{alarm}) =$
  $P(\text{burglary} \mid \text{alarm}) * P(\text{alarm})$
  $= .47 * .19 = .09$
- $P(\text{alarm}) =$
  $P(\text{alarm} \wedge \text{burglary}) +$
  $P(\text{alarm} \wedge \neg\text{burglary})$
  $= .09+.1 = .19$

Example: Inference from the joint

	alarm		¬alarm	
	earthquake	**¬earthquake**	**earthquake**	**¬earthquake**
burglary	.01	.08	.001	.009
¬burglary	.01	.09	.01	.79

P(burglary | alarm) = α P(burglary, alarm)
= α [P(burglary, alarm, earthquake) + P(burglary, alarm, ¬earthquake)
= α [(.01, .01) + (.08, .09)]
= α [(.09, .1)]

Since P(burglary | alarm) + P(¬burglary | alarm) = 1, α = 1/(.09+.1) = 5.26
(i.e., P(alarm) = 1/α = .19 – **quizlet**: how can you verify this?)

P(burglary | alarm) = .09 * 5.26 = .474

P(¬burglary | alarm) = .1 * 5.26 = .526

Exercise: Inference from the joint

p(smart ∧ study ∧ prep)	smart		¬smart	
	study	¬study	study	¬study
prepared	.432	.16	.084	.008
¬prepared	.048	.16	.036	.072

- **Queries:**
  - **What is the prior probability of *smart*?**
  - What is the prior probability of *study*?
  - What is the conditional probability of *prepared*, given *study* and *smart*?

Exercise: Inference from the joint

p(smart ∧ study ∧ prep)	smart		¬smart	
	study	¬study	study	¬study
prepared	.432	.16	.084	.008
¬prepared	.048	.16	.036	.072

- **Queries:**
  - What is the prior probability of *smart*?
  - **What is the prior probability of *study*?**
  - What is the conditional probability of *prepared*, given *study* and *smart*?
- p(smart) = .432 + .16 + .048 + .16 = 0.8

Exercise:
Inference from the joint

p(smart ∧ study ∧ prep)	smart		¬smart	
	study	¬study	study	¬study
prepared	.432	.16	.084	.008
¬prepared	.048	.16	.036	.072

- **Queries:**
  - What is the prior probability of *smart*?
  - What is the prior probability of *study*?
  - **What is the conditional probability of *prepared*, given *study* and *smart*?**
- p(study) = .432 + .048 + .084 + .036 = 0.6

Exercise:
Inference from the joint

p(smart ∧ study ∧ prep)	smart		¬smart	
	study	**¬study**	**study**	**¬study**
prepared	.432	.16	.084	.008
¬prepared	.048	.16	.036	.072

- **Queries:**
  - What is the prior probability of *smart*?
  - What is the prior probability of *study*?
  - What is the conditional probability of *prepared*, given *study* and *smart*?
  - p(prepared | smart, study) = p(prepared, smart, study) / p(smart, study) = .432 / (.432 + .048) = 0.9

Independence

- When variables don't affect each others' probabilities, we call them **independent**, and can easily compute their joint and conditional probability:

  Independent(A, B) $\rightarrow$ $P(A \wedge B) = P(A) * P(B)$, $P(A \mid B) = P(A)$

- {moonPhase, lightLevel} *might* be independent of {burglary, alarm, earthquake}
  - Maybe not: burglars may be more active during a new moon because darkness hides their activity
  - But if we know the light level, the moon phase doesn't affect whether we are burglarized
  - If burglarized, light level doesn't affect if alarm goes off
- Need a more complex notion of independence and methods for reasoning about the relationships

Exercise: Independence

p(smart ∧ study ∧ prep)	smart		¬smart	
	study	**¬study**	**study**	**¬study**
prepared	.432	.16	.084	.008
¬prepared	.048	.16	.036	.072

Queries:

- **Q1: Is *smart* independent of *study*?**
- **Q2: Is *prepared* independent of *study*?**

How can we tell?

Exercise: Independence

p(smart ∧ study ∧ prep)	smart		¬smart	
	study	**¬study**	**study**	**¬study**
prepared	.432	.16	.084	.008
¬prepared	.048	.16	.036	.072

Q1: Is *smart* independent of *study*?

- You might have some intuitive beliefs based on your experience)
- You can check the data

Exercise: Independence

p(smart ∧ study ∧ prep)	smart		¬smart	
	study	¬study	study	¬study
prepared	.432	.16	.084	.008
¬prepared	.048	.16	.036	.072

Q1: Is *smart* independent of *study*?

- Q1 true iff p(smart | study) == p(smart)
  p(smart | study) = p(smart, study) / p(study)
  = (.432 + .048) / .6 = 0.8
  0.8 == 0.8, so smart is independent of study

Exercise: Independence

p(smart ∧ study ∧ prep)	smart		¬smart	
	study	¬study	study	¬study
prepared	.432	.16	.084	.008
¬prepared	.048	.16	.036	.072

Q2: Is *prepared* independent of *study*?

- What is prepared?
- Q2 true iff

Exercise: Independence

p(smart ∧ study ∧ prep)	smart		¬smart	
	study	¬study	study	¬study
prepared	.432	.16	.084	.008
¬prepared	.048	.16	.036	.072

Q2: Is *prepared* independent of *study*?

- Q2 true iff p(prepared | study) == p(prepared)
  p(prepared | study) = p(prepared, study) / p(study)
  = (.432 + .084) / .6 = .86
  0.86 =/= 0.8, so prepared not independent of study

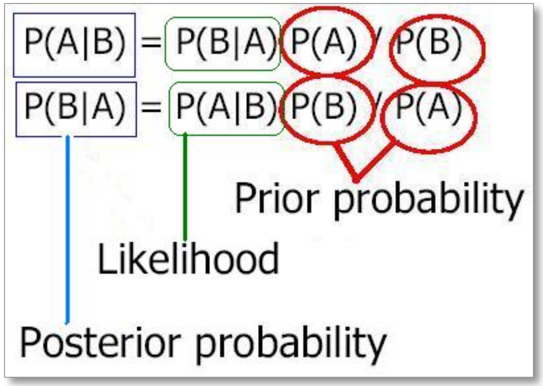
Conditional independence

- Absolute independence:
  - A and B are **independent** if $P(A \wedge B) = P(A) * P(B)$; equivalently, $P(A) = P(A \mid B)$ and $P(B) = P(B \mid A)$
- A and B are **conditionally independent** given C if
  - $P(A \wedge B \mid C) = P(A \mid C) * P(B \mid C)$
- This lets us decompose the joint distribution:
  - $P(A \wedge B \wedge C) = P(A \mid C) * P(B \mid C) * P(C)$
- Moon-Phase and Burglary are ***conditionally independent given*** Light-Level
- Conditional independence is weaker than absolute independence, but useful in decomposing the full joint probability distribution

Conditional independence

- An intuitive understanding is that conditional independence often arises due to causal relations
  - Phase of moon causally effects the level of light at night
  - Other things do too, e.g., presence of street lights
- With respect to our burglary scenario, moon’s phase doesn’t directly effect anything else
- So knowing the lighting level means we can ignore the moon phase in predicting wheter or not an alarm means we had a burglary

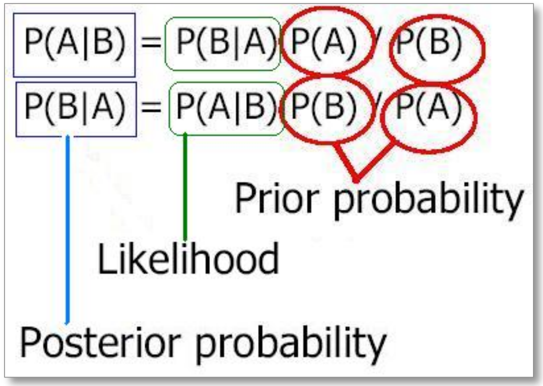
Bayes' rule



- Derived from the product rule:
  - P(C, E) = P(C | E) * P(E)
  - P(E, C)) = P(E | C) * P(C)
  - P(C, E) = P(E, C)

  So…
  - P(C | E) = P(E | C) * P(C) / P(E)

Bayes’ rule



- Derived from the product rule:
  - P(C | E) = P(E | C) * P(C) / P(E)
- Often useful for diagnosis:
  - If E are (observed) effects and C are (hidden) causes,
  - We often have a model for how causes lead to effects P(E | C)
  - We may also have prior beliefs (based on experience) about the frequency of occurrence of causes (P(C))
  - Which allows us to reason abductively from effects to causes (P(C | E))

Ex: meningitis and stiff neck

- Meningitis (M) can cause a a stiff neck (S), though here are many other causes for S, too
- We'd like to use S as a diagnostic symptom and estimate $p(M|S)$
- Studies can easily estimate $p(M)$, $p(S)$ and $p(S|M)$

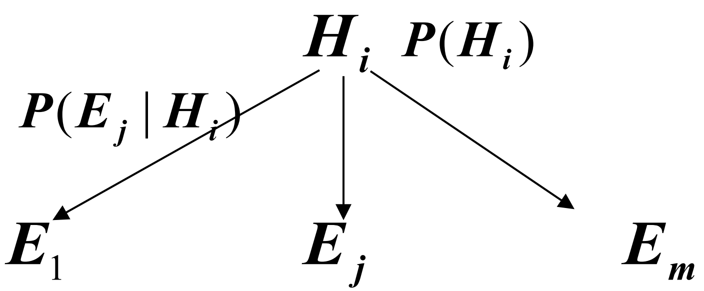
  $p(M)=0.7$, $p(S)=0.01$, $p(M)=0.00002$
- Applying Bayes' Rule:

  $p(M|S) = p(S|M) * p(M) / p(S) = 0.0014$
- We can also do this w/o $p(S)$ if we know $p(S|\sim M)$

  $\alpha <p(S|M)*P(m), p(S|\sim M)*p(\sim M)>$

Bayesian inference

- In the setting of diagnostic/evidential reasoning



hypotheses

evidence/manifestations

$P(H_i)$

– Know prior probability of hypothesis $P(E_j \mid H_i)$

conditional probability $P(H_i \mid E_j)$

– Want to compute the *posterior probability*

- Bayes' s theorem (formula 1):

$$P(H_i \mid E_j) = P(H_i) * P(E_j \mid H_i) / P(E_j)$$

Simple Bayesian diagnostic reasoning

- Also known as: Naive Bayes classifier
- Knowledge base:
  - Evidence / manifestations: $E_1, \ldots E_m$
  - Hypotheses / disorders: $H_1, \ldots H_n$

    Note: E_j and H_i are **binary**; hypotheses are **mutually exclusive** (non-overlapping) and **exhaustive** (cover all possible cases)
  - Conditional probabilities: $P(E_j \mid H_i)$, $i = 1, \ldots n$; $j = 1, \ldots m$
- Cases (evidence for a particular instance): $E_1, \ldots, E_l$
- Goal: Find the hypothesis H_i with the highest posterior
  - $\text{Max}_i \, P(H_i \mid E_1, \ldots, E_l)$

Simple Bayesian diagnostic reasoning

- Bayes' rule says that

  $P(H_i \mid E_1 \ldots E_m) = P(E_1 \ldots E_m \mid H_i)\ P(H_i)\ /\ P(E_1 \ldots E_m)$

- Assume each evidence E_i is conditionally independent of the others, *given* a hypothesis H_i, then:

  $P(E_1 \ldots E_m \mid H_i) = \prod^{m}_{j=1} P(E_j \mid H_i)$

- If we only care about relative probabilities for the H_i, then we have:

  $P(H_i \mid E_1 \ldots E_m) = \alpha\ P(H_i) \prod^{m}_{j=1} P(E_j \mid H_i)$

Limitations

- Can't easily handle multi-fault situations or cases where intermediate (hidden) causes exist:
  - Disease D causes syndrome S, which causes correlated manifestations M_1 and M_2
- Consider composite hypothesis $H_1 \wedge H_2$, where H_1 & H_2 independent. What's relative posterior?

$P(H_1 \wedge H_2 \mid E_1, \ldots, E_l) = \alpha\, P(E_1, \ldots, E_l \mid H_1 \wedge H_2)\, P(H_1 \wedge H_2)$

$= \alpha\, P(E_1, \ldots, E_l \mid H_1 \wedge H_2)\, P(H_1)\, P(H_2)$

$= \alpha \prod_{j=1}^{l} P(E_j \mid H_1 \wedge H_2)\, P(H_1)\, P(H_2)$

- How do we compute $P(E_j \mid H_1 \wedge H_2)$?

Limitations

- Assume H1 and H2 are independent, given E1, …, El?
  - $P(H_1 \wedge H_2 \mid E_1, \ldots, E_l) = P(H_1 \mid E_1, \ldots, E_l)\, P(H_2 \mid E_1, \ldots, E_l)$
- This is a very unreasonable assumption
  - Earthquake and Burglar are independent, but *not* given Alarm:
    - P(burglar | alarm, earthquake) << P(burglar | alarm)
- Another limitation is that simple application of Bayes's rule doesn't allow us to handle causal chaining:
  - A: this year's weather; B: cotton production; C: next year's cotton price
  - A influences C indirectly: $A \rightarrow B \rightarrow C$
  - $P(C \mid B, A) = P(C \mid B)$
- Need a richer representation to model interacting hypotheses, conditional independence, and causal chaining
- Next: conditional independence and Bayesian networks!

Summary

- Probability is a rigorous formalism for uncertain knowledge
- Joint probability distribution specifies probability of every atomic event
- Can answer queries by summing over atomic events
- But we must find a way to reduce the joint size for non-trivial domains
- Bayes' rule lets unknown probabilities be computed from known conditional probabilities, usually in the causal direction
- Independence and conditional independence provide tools

Postscript: Frequentists vs. Bayesians

- Frequentist inference draws conclusions from sample data based on the frequency or proportion of the data
- Bayesian inference uses Bayes' rule to update probability estimates for a hypothesis as additional evidence is learned
- The differences are often subtle, but can be consequential

Frequentists vs. Bayesians

http://xkcd.com/1132/